EQUATIONS FOR COMPRESSIBLE CONVECTION IN RAPIDLY ROTATING SPHERICAL SHELLS (ANELASTIC APPROXIMATION)

1. Six Input parameters

$$Ra = \frac{GMd\Delta S}{\nu\kappa c_p}, \qquad Pr = \frac{\nu}{\kappa}, \qquad E = \frac{\nu}{2\Omega d^2}$$

$$N_\rho = \ln\left(\frac{\rho_i}{\rho_o}\right), \qquad n, \text{ polytropic index}, \qquad \eta = \frac{r_i}{r_o} \tag{1a-f}$$

Here in dimensional units $G = 6.67 \times 10^{-11}$ is gravitational constant,$M = 1.9 \times 10^{27}$ is mass of Jupiter, $r_O = 7 \times 10^7$ radius of outer shell, $r_i = 5.6 \times 10^7$ radius of inner shell. $d = r_o - r_i$, ΔS is entropy drop from r_i to r_o, ν is eddy kinematic viscosity, assumed constant across shell, κ is eddy entropy diffusivity, also assumed constant across shell. c_p is specific heat at constant pressure. $\Omega = 1.76 \times 10^{-4}$ is rotation rate.

2. Dimensionless units

Length scale d, timescale d/ν^2, Mass $\rho_0 d^3$, where ρ_0 is density at $\zeta = 1$ (see below), unit of Entropy $Pr\Delta S$. Note factor of Pr in entropy scaling, so entropy on r_i is fixed at Pr^{-1}.

3. Polytropic density distribution in the dimensionless units

$$c_0 = \frac{2\zeta_0 - \eta - 1}{1-\eta}, \qquad c_1 = \frac{(1+\eta)(1-\zeta_0)}{(1-\eta)^2}, \qquad \zeta_0 = \frac{\eta+1}{\eta\exp(N_\rho/n)+1}.$$

$$\rho = \zeta^n, \qquad T_a = \zeta, \qquad \zeta = c_0 + \frac{c_1}{r}, \qquad \xi = \frac{n}{\zeta}\frac{d\zeta}{dr}, \qquad r_i = \frac{\eta}{1-\eta} < r < r_o = \frac{1}{1-\eta} \tag{2a-h}$$

T_a is the dimensionless basic state adiabatic temperature. ρ is the dimensionless basic state density. The basic state is programmed in subroutine mes_precompute in module meshs.f90. Both ξ and $d\xi/dr$ are required, and are stored as xi(:,0:1), $d\xi/dr$ being xi(:,1).

4. Dimensionless equation of motion

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u}\times\boldsymbol{\omega} - E^{-1}\boldsymbol{\Omega}\times\mathbf{u} - \nabla\left(\frac{p'}{\rho} + \frac{1}{2}\mathbf{u}^2\right) + \mathbf{F}_\nu + \frac{RaS}{r^2}\hat{\mathbf{r}} \tag{3}$$

where

$$\boldsymbol{\omega} = \nabla\times\mathbf{u} \qquad \mathbf{F}_\nu = \frac{1}{\rho}\frac{\partial}{\partial x_j}\rho\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3\rho}\frac{\partial}{\partial x_i}\rho\frac{\partial u_j}{\partial x_j} \tag{4a-b}$$

where $\mathbf{u}$ is velocity, $\boldsymbol{\Omega}$ angular rotation vector, S is the entropy.
Let

$$\mathbf{F}_\nu = \hat{\mathbf{F}}_\nu - \frac{1}{\rho}\nabla\times\nabla\times\rho\mathbf{u}, \tag{5}$$

and

$$\hat{\mathbf{F}}_\nu = \frac{1}{3}\nabla(\nabla\cdot\mathbf{u}) + \frac{1}{\rho}(\nabla\rho\times\boldsymbol{\omega}) - \frac{\mathbf{u}}{\rho}\nabla^2\rho + \frac{2}{3}\hat{\mathbf{r}}u_r\frac{1}{\rho^2}\left(\frac{d\rho}{dr}\right)^2. \tag{6}$$

The first term of this can be ignored since it vanishes when we take the curl of the equation of motion.

$$\frac{1}{\rho}(\nabla\rho\times\boldsymbol{\omega}) = -\xi\omega_\phi\hat{\hat{\boldsymbol{\theta}}} + \xi\omega_\theta\hat{\boldsymbol{\phi}}, \quad \frac{\mathbf{u}}{\rho}\nabla^2\rho = \mathbf{u}\left(\frac{2}{r}\xi + \frac{d\xi}{dr} + \xi^2\right), \quad \frac{2}{3}\hat{\mathbf{r}}u_r\frac{1}{\rho^2}\left(\frac{d\rho}{dr}\right)^2 = \frac{2}{3}\hat{\mathbf{r}}u_r\xi^2. \quad (7a-c)$$

We write the equation of motion in the form

$$\frac{\partial\mathbf{u}}{\partial t} + \frac{1}{\rho}\nabla\times\nabla\times\rho\mathbf{u} = -\nabla\hat{p} + \mathbf{N}_v \tag{8}$$

where

$$\mathbf{N}_v = \mathbf{u}\times\boldsymbol{\omega} + \hat{\mathbf{F}}_\nu - E^{-1}\boldsymbol{\Omega}\times\mathbf{u} + \frac{RaS}{r^2}\hat{\mathbf{r}}, \tag{9}$$

$\hat{\mathbf{F}}_\nu$ being given by (6) and (7). The terms in equation (9) are coded in this order in routine non_velocity in module nonlinear.

5. Dimensionless continuity equation

$$\nabla\cdot\rho\mathbf{u} = 0, \tag{10}$$

which implies

$$\mathbf{u} = \frac{1}{\rho}\nabla\times\nabla\times\mathbf{r}P\rho + \frac{1}{\rho}\nabla\times\mathbf{r}T\rho, \tag{11}$$

noting carefully that $\mathbf{r}$ and not $\hat{\mathbf{r}}$ appears in equation (5).

6. Dimensionless entropy equation

$$Pr\frac{\partial S}{\partial t} = -Pr\mathbf{u}\cdot\nabla S + \zeta^{-n-1}\nabla\cdot\zeta^{n+1}\nabla S + \frac{Di}{\zeta}Q_v \tag{12}$$

where

$$Di = \frac{c_1 Pr}{Ra} = \frac{\eta(1+\eta)Pr}{(1-\eta)^2 Ra}\frac{(\exp(N_\rho/n)-1)}{(1+\eta\exp(N_\rho/n))} \tag{13}$$

$$Q_v = 2\left[e_{ij}e_{ij} - \frac{1}{3}(\nabla\cdot\mathbf{u})^2\right], \qquad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (14a-b)$$

This is solved by writing

$$Pr\frac{\partial S}{\partial t} - \frac{1}{\rho}\nabla\cdot\rho\nabla S = Pr\frac{\partial S}{\partial t} - \frac{\partial^2 S}{\partial r^2} - \left(\frac{2}{r}+\xi\right)\frac{\partial S}{\partial r} + \frac{\ell(\ell+1)S}{r^2} = -Pr\mathbf{u}\cdot\nabla S + \frac{\xi}{n}\frac{\partial S}{\partial r} + \frac{Di}{\zeta}Q_v. \tag{15}$$

The terms on the right hand side of (15) are programmed into subroutine non_codensity, in module nonlinear.f90. The terms on the left are calculated in tim_lumesh_X and tim_mesh_Y in subroutine timestep.f90. The radial derivative part is computed in subroutine mes_rdom_init in module meshs.f90 and is stored as mes_oc%GreLap. The name is because Entlap is the same as GreLap, see section 7 below. Note that tim_lumesh_X and tim_mesh_Y are set up in subroutine cod_matrices in module codensity.f90, which in turn is called by routine initialise in module main.f90.
Clune et al. gives

$$2\left[e_{ij}e_{ij} - \frac{1}{3}(\nabla\cdot\mathbf{u})^2\right] = 2\left(\frac{\partial u_r}{\partial r}\right)^2 + 2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right)^2 - \frac{2}{3}(\nabla\cdot\mathbf{u})^2 + 2\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cos\theta}{r\sin\theta}\right)^2$$

$$+\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}\right)^2+\left(\frac{\partial u_\phi}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r}\right)^2+\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}-\frac{u_\phi\cos\theta}{r\sin\theta}\right)^2 \quad (16)$$

Split this into

$$Q_v = Q_v^{(1)} + Q_v^{(2)}, \quad (17)$$

where

$$Q_v^{(1)} = 2\left(\frac{\partial u_r}{\partial r}\right)^2+2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)^2-\frac{2}{3}\left(\nabla\cdot\mathbf{u}\right)^2+2\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}+\frac{u_r}{r}+\frac{u_\theta\cos\theta}{r\sin\theta}\right)^2 \quad (18)$$

$$Q_v^{(2)} = \left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}\right)^2+\left(\frac{\partial u_\phi}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r}\right)^2+\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}-\frac{u_\phi\cos\theta}{r\sin\theta}\right)^2 \quad (19)$$

Define

$$q_v^{(1)} = \frac{u_\theta\cos\theta}{r\sin\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}, \qquad q_v^{(2)} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}-\frac{u_\phi\cos\theta}{r\sin\theta}, \quad (20a-b)$$

and we can write

$$Q_v^{(1)} = 2\left(\frac{\partial u_r}{\partial r}\right)^2-\frac{2}{3}(\xi u_r)^2+2\left(\frac{\partial u_r}{\partial r}+u_r(\xi+\frac{1}{r})+q_v^{(1)}\right)^2+2\left(q_v^{(1)}+\frac{u_r}{r}\right)^2, \quad (21)$$

and recalling $\boldsymbol{\omega} = \nabla\times\mathbf{u}$,

$$Q_v^{(2)} = \left(2\frac{\partial u_\theta}{\partial r}-\omega_\phi\right)^2+\left(2\frac{\partial u_\phi}{\partial r}+\omega_\theta\right)^2+\left(2q_v^{(2)}+\omega_r\right)^2 \quad (22)$$

Since the components of $\boldsymbol{\omega}$ and $\mathbf{u}$ are all required, evaluated on the physical mesh, for the convective derivative, the only new terms required on the mesh are the radial derivatives of $\mathbf{u}$, $q_v^{(1)}$ and $q_v^{(2)}$. These are all calculated in subroutine tra_qst2rtp_new in module transform.fftw3.f90, which is called by vel_transform in module velocity.f90.

qst form of the variables

We need the qst form of the variables for this evaluation. This is defined by

$$u_r = q(r)Y_\ell^m, \quad q(r) = \ell(\ell+1)\frac{P}{r}, \quad (23a-b)$$

$$u_\theta = \frac{1}{\sin\theta}\frac{\partial T}{\partial \phi}+\frac{1}{r\rho}\frac{\partial}{\partial r}r\rho\frac{\partial P}{\partial \theta} = -\frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{\sin\theta}\frac{\partial t}{\partial \phi}+\frac{1}{\sqrt{\ell(\ell+1)}}\frac{\partial s}{\partial \theta}, \quad (24)$$

$$u_\phi = -\frac{\partial T}{\partial \theta}+\frac{1}{r\rho}\frac{\partial}{\partial r}r\rho\frac{\partial P}{\partial \phi} = \frac{1}{\sqrt{\ell(\ell+1)}}\frac{\partial t}{\partial \theta}+\frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{\sin\theta}\frac{\partial s}{\partial \phi} \quad (25)$$

Since

$$t = -\sqrt{\ell(\ell+1)}T, \quad s = \sqrt{\ell(\ell+1)}\frac{1}{r\rho}\frac{\partial}{\partial r}(r\rho P) \quad (26a-b)$$

where P and T are the poloidal and toroidal scalars defined in (11). Note that q, s and t have these very simple expressions in terms of P and T, but (26b) involves a radial derivative of the density, so subroutine var_coll_TorPol2qst in module variables.f90 has been modified to add in a compressible term, and a new routine to evaluate the radial derivatives of q,s and t, var_coll_TorPol2qstderiv, has been added to module variables.f90.

The radial derivative terms are exactly analogous to the ordinary terms.

$$q_v^{(1)} = \frac{1}{\sqrt{\ell(\ell+1)}}\frac{\cos\theta}{r\sin\theta}\frac{\partial s}{\partial\theta} + \frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{r\sin^2\theta}\frac{\partial^2 s}{\partial\phi^2}$$

$$+\frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{r\sin\theta}\frac{\partial^2 t}{\partial\theta\partial\phi} - \frac{1}{\sqrt{\ell(\ell+1)}}\frac{\cos\theta}{r\sin^2\theta}\frac{\partial t}{\partial\phi} \tag{27}$$

$$q_v^{(2)} = \frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{r\sin\theta}\frac{\partial^2 s}{\partial\theta\partial\phi} - \frac{1}{\sqrt{\ell(\ell+1)}}\frac{\cos\theta}{r\sin^2\theta}\frac{\partial s}{\partial\phi}$$

$$-\frac{1}{\sqrt{\ell(\ell+1)}}\frac{\cos\theta}{r\sin\theta}\frac{\partial t}{\partial\theta} - \frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{r\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2} \tag{28}$$

7. Equations for P and T

$$\frac{\partial T}{\partial t} - \frac{1}{\rho}\nabla^2(\rho T) = \frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial r^2} - \left(\frac{2}{r} + 2\xi\right)\frac{\partial T}{\partial r} - \left(\frac{2\xi}{r} + \xi^2 + \frac{d\xi}{dr}\right)T + \frac{\ell(\ell+1)T}{r^2}$$

$$= \frac{r}{\ell(\ell+1)}\hat{\mathbf{r}}\cdot\nabla\times\mathbf{N}_v \tag{29}$$

$$\frac{\partial P}{\partial t} - \frac{1}{\rho}\nabla^2(\rho P) = \frac{\partial P}{\partial t} - \frac{\partial^2 P}{\partial r^2} - \left(\frac{2}{r} + 2\xi\right)\frac{\partial P}{\partial r} - \left(\frac{2\xi}{r} + \xi^2 + \frac{d\xi}{dr}\right)P + \frac{\ell(\ell+1)P}{r^2} = \hat{G} \tag{30}$$

$$\nabla^2\hat{G} + \frac{1}{r}\frac{\partial}{\partial r}(r\xi\hat{G}) = \frac{\partial^2\hat{G}}{\partial r^2} + \left(\frac{2}{r} + \xi\right)\frac{\partial\hat{G}}{\partial r} + \left(\frac{\xi}{r} + \frac{d\xi}{dr}\right)\hat{G} + \frac{\ell(\ell+1)\hat{G}}{r^2}$$

$$= -\frac{r}{\ell(\ell+1)}\hat{\mathbf{r}}\cdot\nabla\times\nabla\times\mathbf{N}_v \tag{31}$$

Here $\hat{G}$ is the Green's function for the poloidal equation.
The right hand-sides are computed in subroutine non_velocity in module nonlinear.f90. The curls are computed in subroutines var_coll_qstllcurlr and var_coll_qstllcurlcurlr in module variables.f90. The factor $r/\ell(\ell+1)$ is included in var routines. The terms on the left are calculated in tim_lumesh_vel_X and tim_mesh_vel_Y in subroutine timestep.f90. The radial derivative part is computed in subroutine mes_rdom_init in module meshs.f90 and is stored as mes_oc%CompLap. Note that tim_lumesh_vel_X and tim_mesh_vel_Y are set up in subroutine vel_matrices in module velocity.f90, which in turn is called by routine initialise in module main.f90.

8. Boundary conditions

Entropy

$$S = 1/Pr \quad \text{on} \quad r = r_i, \quad S = 0 \quad \text{on} \quad r = r_o. \tag{32}$$

These boundary conditions are set in subroutine cod_setbc in module codensity.f90.
Toroidal velocity scalar applied on $r = r_i$ or $r = r_o$ as appropriate,

$$\frac{\partial T}{\partial r} - \frac{T}{r} = 0 \quad \mathrm{stress-free}, \quad T = 0 \quad \mathrm{no-slip}. \tag{33}$$

Poloidal velocity scalar applied on $r = r_i$ or $r = r_o$ as appropriate,

$$\frac{\partial^2 P}{\partial r^2} + \xi \frac{\partial P}{\partial r} = 0 \quad \mathrm{stress-free}, \qquad \frac{\partial P}{\partial r} = 0 \quad \mathrm{no-slip}. \tag{34}$$

These assume no-penetration $u_r = 0$ at the boundaries, i.e $P = 0$. This condition is automatically imposed by the Green's function method for solving (31). The velocity boundary conditions are set in vel_bc_Tor and vel_bc_Pol in module velocity.f90.

9. Energy Balance

Multiply (12) by ζ^{n+1} and integrate over the volume of the shell.

$$\frac{\partial}{\partial t}\int \zeta^{n+1} Pr S \, dv = -Pr \int \zeta^{n+1}(\mathbf{u}\cdot\nabla)S \, dv + \int \nabla\cdot(\zeta^{n+1}\nabla S)\, dv + \int \zeta^n Di Q_v \, dv \tag{35}$$

Now

$$-Pr \int \zeta^{n+1}(\mathbf{u}\cdot\nabla)S \, dv = -\int \nabla\cdot(Pr\zeta^{n+1}\mathbf{u}S)\, dv + \int Pr\nabla\cdot(\zeta^n\mathbf{u})\zeta S \, dv + \int PrS\zeta^n(\mathbf{u}\cdot\nabla)\zeta \, dv$$

$$= -Prc_1 \int \zeta^n S \frac{u_r}{r^2}\, dv, \tag{36}$$

using the divergence theorem, the continuity equation (10), and 2(c). So

$$\frac{\partial}{\partial t}\int \zeta^{n+1} Pr S \, dv = -Prc_1 \int \zeta^n S\frac{u_r}{r^2}\, dv - \int_{S_i} \zeta_i^{n+1}\frac{\partial S}{\partial r}\, r_i^2 \sin\theta \, d\theta d\phi$$

$$+ \int_{S_o} \zeta_o^{n+1}\frac{\partial S}{\partial r}\, r_o^2 \sin\theta \, d\theta d\phi + Di \int \zeta^n Q_v \, dv \tag{37}$$

Now multiply the equation of motion (3) by $\zeta^n \mathbf{u} c_1 Pr/Ra$ and integrate over the volume between the shells.

$$\frac{c_1 Pr}{Ra}\frac{\partial}{\partial t}\int \frac{1}{2}\zeta^n \mathbf{u}^2 \, dv = \int \frac{c_1 Pr}{Ra}\zeta^n \mathbf{u}\cdot\mathbf{F}_\nu + c_1 Pr \int \zeta^n S \frac{u_r}{r^2}\, dv, \tag{38}$$

using the divergence theorem to remove the pressure term.
Now adding (37) and (38) we get the dimensionless global energy balance equation

$$\frac{\partial}{\partial t}\int \zeta^{n+1} Pr S + \frac{c_1 Pr}{2Ra}\zeta^n \mathbf{u}^2 \, dv = -\int_{S_i} \zeta_i^{n+1}\frac{\partial S}{\partial r}\, r_i^2 \sin\theta \, d\theta d\phi + \int_{S_o} \zeta_o^{n+1}\frac{\partial S}{\partial r}\, r_o^2 \sin\theta \, d\theta d\phi. \tag{39}$$

To see that the dissipation terms in (37) and (38) do indeed cancel, note $Di = c_1 Pr/Ra$ from (13) and

$$\int \zeta^n \mathbf{u}\cdot\mathbf{F}_\nu \, dv = -\int \zeta^n \frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\zeta^n(\nabla\cdot\mathbf{u})^2 \, dv \tag{40}$$

from 4(b) and the divergence theorem. Now from 14(b)

$$2e_{ij}e_{ij} = \frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{41}$$

so using (14)

$$\int \zeta^n Q_v \, dv = -\int \zeta^n \mathbf{u}\cdot\mathbf{F}_\nu \, dv. \tag{42}$$

9. Linearised equations and the Basic State

If $\mathbf{u} = 0$ and S is steady, then (12) implies

$$\nabla \cdot \zeta^{n+1} \nabla \bar{S} = 0, \tag{42}$$

and the solution satisfying the boundary conditions (32) is

$$\bar{S} = \frac{Pr^{-1}(\zeta_o^{-n} - \zeta^{-n})}{\zeta_o^{-n} - \zeta_i^{-n}}. \tag{43}$$

Note that

$$\zeta_i = c_0 + \frac{c_1}{r_i}, \quad \zeta_o = c_0 + \frac{c_1}{r_o} \tag{44}$$

Writing $S = \bar{S} + S'$, the linear equations are then

$$Pr \frac{\partial S'}{\partial t} = -Pr \mathbf{u} \cdot \nabla \bar{S} + \zeta^{-n-1} \nabla \cdot \zeta^{n+1} \nabla S', \tag{45}$$

or equivalently

$$Pr \frac{\partial S'}{\partial t} = \frac{\xi \zeta^{-n}}{\zeta_o^{-n} - \zeta_i^{-n}} u_r + \zeta^{-n-1} \nabla \cdot \zeta^{n+1} \nabla S', \tag{46}$$

The linearised equation of motion being

$$\frac{\partial \mathbf{u}}{\partial t} = -E^{-1} \mathbf{\Omega} \times \mathbf{u} - \nabla \left(\frac{p'}{\rho} \right) + \mathbf{F}_\nu + \frac{RaS'}{r^2} \hat{\mathbf{r}}. \tag{47}$$

10. Nusselt number

The Nusselt number is the ratio of the heat conducted in at the bottom divided by the heat conducted in by the basic state. It is also the ratio of the heat conducted out at the top divided by the heat conducted out by the basic state. From (39) these must be equal in a steady state, i.e. equal when averaged over time.

$$Nu_{bot} = \frac{\int_{S_i} \zeta_i^{n+1} \frac{\partial S}{\partial r} r_i^2 \sin\theta \, d\theta d\phi}{\int_{S_i} \zeta_i^{n+1} \frac{\partial \bar{S}}{\partial r} r_i^2 \sin\theta \, d\theta d\phi}, \quad Nu_{top} = \frac{\int_{S_o} \zeta_o^{n+1} \frac{\partial S}{\partial r} r_o^2 \sin\theta \, d\theta d\phi}{\int_{S_o} \zeta_o^{n+1} \frac{\partial \bar{S}}{\partial r} r_o^2 \sin\theta \, d\theta d\phi}, \tag{48}$$

Using (43), these can be written,

$$Nu_{bot} = -\frac{(\exp N_\rho - 1)\zeta_i r_i^2 Pr}{4\pi n c_1} \int_{S_i} \frac{\partial S}{\partial r} \sin\theta \, d\theta d\phi,$$

$$Nu_{top} = -\frac{(1 - \exp -N_\rho)\zeta_o r_o^2 Pr}{4\pi n c_1} \int_{S_o} \frac{\partial S}{\partial r} \sin\theta \, d\theta d\phi, \tag{49}$$

or alternatively

$$Nu_{bot} = -\frac{(\exp N_\rho - 1)Pr}{4\pi \xi_i} \int_{S_i} \frac{\partial S}{\partial r} \sin\theta \, d\theta d\phi,$$

$$Nu_{top} = -\frac{(1 - \exp(-N_\rho))Pr}{4\pi \xi_o} \int_{S_o} \frac{\partial S}{\partial r} \sin\theta \, d\theta d\phi, \tag{50}$$

ξ_i and ξ_o being the values on the appropriate boundaries.