Physica D 170 (2002) 253–286
PHYSICA D
www.elsevier.com/locate/physd

Averaged Lagrangians and the mean effects of fluctuations in ideal fluid dynamics

Darryl D. Holm

Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, MS B284, Los Alamos, NM 87545, USA

Received 20 February 2001; received in revised form 14 May 2002; accepted 31 May 2002
Communicated by I. Gabitov

Dedicated to Rupert Ford (1968–2001)

Abstract

We begin by placing the generalized Lagrangian mean (GLM) equations for a compressible adiabatic fluid into the Euler–Poincaré (EP) variational framework of fluid dynamics, for an averaged Lagrangian. We then state the EP Averaging Result—that GLM equations arise from GLM Hamilton's principles in the EP framework. Next, we derive a new set of approximate small-amplitude GLM equations ($g\ell m$ equations) at second order in the fluctuating displacement of a Lagrangian trajectory from its mean position. These equations express the linear and nonlinear back-reaction effects on the Eulerian mean fluid quantities by the fluctuating displacements of the Lagrangian trajectories in terms of their Eulerian second moments. The derivation of the $g\ell m$ equations uses the linearized relations between Eulerian and Lagrangian fluctuations, in the tradition of Lagrangian stability analysis for fluids. The $g\ell m$ derivation also uses the method of averaged Lagrangians, in the tradition of wave, mean flow interaction (WMFI). The $g\ell m$ EP motion equations for compressible and incompressible ideal fluids are compared with the Euler-alpha turbulence closure equations. An alpha model is a GLM (or $g\ell m$) fluid theory with a Taylor hypothesis closure (THC). Such closures are based on the linearized fluctuation relations that determine the dynamics of the Lagrangian statistical quantities in the Euler-alpha closure equations. We use the EP Averaging Result to bridge between the GLM equations and the Euler-alpha closure equations. Hence, combining the small-amplitude approximation with THC yields in new $\overline{g\ell m}$ turbulence closure equations for compressible fluids in the EP variational framework.

Keywords: Averaged Lagrangians; Ideal fluid dynamics; Taylor hypothesis; Turbulence closures

1. Brief review of generalized Lagrangian mean (GLM) theory for compressible fluids

An exceptional accomplishment in formulating averaged motion equations for fluid dynamics is the GLM theory of nonlinear waves on a Lagrangian-mean flow, as explained in two consecutive papers of Andrews and McIntyre [2,3]. This section introduces the results that we shall need later from the rather complete description given in these papers. Even now, these fundamental papers still make worthwhile reading and are taught in many atmospheric science departments. Section 2 begins by placing the GLM equations for a rotating adiabatic compressible fluid into the Euler–Poincaré (EP) variational framework of fluid dynamics in the Eulerian description. This is first done

E-mail address: dholm@lanl.gov (D.D. Holm).

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PII: S0167-2789(02)00552-3

explicitly by re-deriving the GLM equations using Hamilton's principle with a GLM averaged Lagrangian. We then explain that every ideal GLM continuum equation follows from a GLM averaged variational principle, via the Lagrangian averaged Euler–Poincaré (LAEP) theorem of Holm [22,23]. The LAEP theorem also translates the GLM theory into the geometrical language of EP theory. However, in this paper we shall stay with the classical vector notation of fluid dynamics.

In Section 3, we recall the standard linearized Eulerian/Lagrangian fluctuation relations. This is done in preparation for Section 4 in which we construct a small-amplitude approximation of the GLM equations for a compressible adiabatic fluid at second order in the fluctuating displacement of a Lagrangian trajectory from its mean position. Substituting the *linear* fluctuation relations into the GLM action principle in the EP framework turns out to have both linear *and* nonlinear effects on the resulting EP equations. Another characteristic feature of these small-amplitude GLM equations ($g\ell m$ equations) is that they involve second-gradients of Eulerian mean flow quantities, in combination with quadratic moments of the Lagrangian displacement statistics. The latter must be modeled in closing the system.

In Section 5, we introduce several additional modeling options that can produce second-order closures $\overline{g\ell m}$ for the $g\ell m$ equations. These modeling options are formulated as variants of the Taylor hypothesis for frozen-in turbulence. The resulting Taylor hypothesis closure (THC) models recover the recently discovered Euler-alpha equations for incompressible ideal fluids. The THC models also provide a systematic basis for extending the Euler-alpha closure equations to the *compressible case*. Thus, the Euler-alpha equations reappear in a more general context than in their original derivation in [25,26]. Section 6 summarizes these THC results and provides a synopsis of the paper.

1.1. GLM motion equation for adiabatic compressible fluids

The GLM equations are based on defining fluid quantities at a displaced fluctuating position. In the GLM description, $\bar{\chi}$ denotes the Eulerian mean of a fluid quantity $\chi = \bar{\chi} + \chi'$, while $\bar{\chi}^{\rm L}$ denotes the Lagrangian mean of the same quantity, defined by

$$\bar{\chi}^{\rm L}(\mathbf{x}) \equiv \overline{\chi^{\xi}(\mathbf{x})}, \quad \text{with} \;\; \chi^{\xi}(\mathbf{x}) \equiv \chi(\mathbf{x} + \xi(\mathbf{x}, t)). \tag{1.1}$$

Here $\mathbf{x}^{\xi} \equiv \mathbf{x} + \xi(\mathbf{x}, t)$ is the current position of a Lagrangian fluid trajectory whose mean position is $\mathbf{x}$. Thus, $\xi(\mathbf{x}, t)$ with vanishing Eulerian mean $\bar{\xi} = 0$ denotes the fluctuating displacement of a Lagrangian particle trajectory about its mean position $\mathbf{x}$. From its defining relation (1.1), one sees that the Lagrangian mean has *two disadvantages*, relative to the Eulerian mean: it is history dependent and it does not commute with the spatial gradient. However, the Lagrangian mean does commute with the advective derivative.

In GLM theory, the difference $\chi^{\xi} - \bar{\chi}^{\rm L} = \chi^{\ell}$ is called the *Lagrangian disturbance* of the quantity χ. One finds $\overline{\chi^{\ell}} = 0$, since the Eulerian mean possesses the *projection property* $\bar{\bar{\chi}} = \bar{\chi}$ for any quantity χ (and, in particular, it possesses that property for χ^{ξ}).[1]

Andrews and McIntyre [2] show that, provided the smooth map $\mathbf{x} \to \mathbf{x} + \xi(\mathbf{x}, t)$ is invertible (i.e., provided the vector field $\xi(\mathbf{x}, t)$ generates a diffeomorphism), then the Lagrangian disturbance velocity $\mathbf{u}^{\ell}$ may be expressed in terms of ξ by

$$\mathbf{u}^{\ell} = \mathbf{u}^{\xi} - \bar{\mathbf{u}}^{\rm L} = \frac{{\rm D}^{\rm L}\xi}{{\rm D}t}, \quad \text{where} \;\; \frac{{\rm D}^{\rm L}\xi}{{\rm D}t} \equiv \frac{\partial \xi}{\partial t} + \bar{\mathbf{u}}^{\rm L} \cdot \nabla \xi.$$

Consequently, the Lagrangian disturbance velocity $\mathbf{u}^{\ell}$ is a genuine fluctuation quantity satisfying $\overline{\mathbf{u}^{\ell}} = 0$, since $\overline{\mathbf{u}^{\xi}} - \overline{\bar{\mathbf{u}}^{\rm L}} = \overline{\mathbf{u}^{\xi}} - \overline{\overline{\mathbf{u}^{\xi}}} = 0$, by the projection property of the Eulerian mean. (Alternatively, $\overline{\mathbf{u}^{\ell}} = \overline{{\rm D}^{\rm L}\xi/{\rm D}t} = 0$ also follows, since the Eulerian mean commutes with ${\rm D}^{\rm L}/{\rm D}t$ and ξ has mean zero.)

[1] Note that spatial filtering in general does *not* possess the projection property.

1.1.1. GLM scalar advection relations

At position $\mathbf{x}$, the velocity $\mathbf{u}^\xi = \bar{\mathbf{u}}^{\mathrm{L}} + \mathbf{u}^\ell$ is the sum of the Lagrangian-mean velocity $\bar{\mathbf{u}}^{\mathrm{L}}$ and the Lagrangian disturbance velocity $\mathbf{u}^\ell$. Thus, $\mathbf{u}^\xi = \mathrm{D}^{\mathrm{L}}\mathbf{x}^\xi/\mathrm{D}t$ and for any scalar field $\chi(\mathbf{x}, t)$ one has

$$\left(\frac{\mathrm{D}\chi}{\mathrm{D}t}\right)^\xi = \frac{\mathrm{D}^{\mathrm{L}}}{\mathrm{D}t}(\chi^\xi).$$

The velocity $\bar{\mathbf{u}}^{\mathrm{L}}$ appearing in the advection operator $\mathrm{D}^{\mathrm{L}}/\mathrm{D}t = \partial_t + \bar{\mathbf{u}}^{\mathrm{L}} \cdot \nabla$ is a mean quantity; so one finds, as expected, that the Lagrangian-mean $\overline{(\cdot)}^{\mathrm{L}}$ commutes with the advective derivative $\mathrm{D}/\mathrm{D}t$. Namely,

$$\left(\overline{\frac{\mathrm{D}\chi}{\mathrm{D}t}}\right)^{\mathrm{L}} = \frac{\mathrm{D}^{\mathrm{L}}}{\mathrm{D}t}(\bar{\chi}^{\mathrm{L}}), \quad \text{and} \quad \left(\frac{\mathrm{D}\chi}{\mathrm{D}t}\right)^\ell = \frac{\mathrm{D}^{\mathrm{L}}}{\mathrm{D}t}\chi^\ell, \tag{1.2}$$

where $\chi^\ell = \chi^\xi - \bar{\chi}^{\mathrm{L}}$ is the Lagrangian disturbance of χ satisfying $\overline{\chi^\ell} = 0$. For example, in an adiabatic compressible flow, the specific entropy s is advected as a scalar. That is, it satisfies $\mathrm{D}s/\mathrm{D}t = 0$ and, consequently, $\mathrm{D}^{\mathrm{L}}\bar{s}^{\mathrm{L}}/\mathrm{D}t = 0$, as well. Hence, $s^\xi = \bar{s}^{\mathrm{L}}$ follows, by integration of $\mathrm{D}^{\mathrm{L}}(\bar{s}^{\mathrm{L}} - s^\xi)/\mathrm{D}t = 0$ along mean trajectories and invertibility of the map $\mathbf{x} \to \mathbf{x} + \xi(\mathbf{x}, t)$.

1.1.2. Mass conservation: the GLM continuity equation

Remarkably, $\bar{D}^{\mathrm{L}}$ is not the density advected in the GLM theory. That is,

$$\partial_t \bar{D}^{\mathrm{L}} + \operatorname{div} \bar{D}^{\mathrm{L}}\bar{\mathbf{u}}^{\mathrm{L}} \neq 0.$$

Instead, GLM satisfies another density advection relation—the *GLM continuity equation*,

$$\partial_t \tilde{D} + \operatorname{div} \tilde{D}\bar{\mathbf{u}}^{\mathrm{L}} = 0, \tag{1.3}$$

for a density $\tilde{D}$, which is also a mean quantity. That is, $\bar{\tilde{D}} = \tilde{D}$, where one invokes the projection property of the Eulerian mean. The GLM conserved density $\tilde{D}$ is given by

$$\tilde{D} = D^\xi \mathcal{J}, \quad \text{where } \mathcal{J} = \det(\nabla_{\mathbf{x}}(\mathbf{x} + \xi)). \tag{1.4}$$

The GLM continuity equation for the density $\tilde{D}$ may be shown by transforming the instantaneous mass conservation relation $D^\xi\, \mathrm{d}^3x^\xi = D(x_0)\, \mathrm{d}^3x_0$ into

$$D^\xi \mathcal{J} \equiv D^\xi(x) \det(\nabla_{\mathbf{x}}(\mathbf{x} + \xi)) = \frac{D(x_0)\, \mathrm{d}^3x_0}{\mathrm{d}^3x} \equiv \tilde{D}$$

and then using the defining relation (1.1) for the Lagrangian mean in terms of the Eulerian mean. In taking the Eulerian mean of this relation, we keep in mind that $\mathbf{x}$ is the mean position, so the right-hand side is *already* an average quantity. Thus, $\tilde{D} = D^\xi \mathcal{J}$ satisfies $\bar{\tilde{D}} = \tilde{D}$, as claimed, and we note that $\tilde{D} \neq \bar{D}^{\mathrm{L}}$, in general. The mean mass conservation relation $\tilde{D}\, \mathrm{d}^3x = D(x_0)\, \mathrm{d}^3x_0$, then implies the continuity equation (1.3) for $\tilde{D}$, upon recalling that $\bar{\mathbf{u}}^{\mathrm{L}}$ is the velocity tangent to the mean Lagrangian position $\mathbf{x}$.

1.1.3. GLM motion equation

Although the Eulerian mean commutes with spatial gradients, it interferes with the advection operator and fails to produce a closed system of equations. In contrast, the Lagrangian mean commutes with the advection operator and produces the following GLM motion equation for adiabatic compressible fluids in a frame rotating with constant frequency Ω, as given in [2] in Cartesian coordinates,

$$\frac{\mathrm{D}^{\mathrm{L}}}{\mathrm{D}t}(\bar{\mathbf{u}}^{\mathrm{L}} - \bar{\mathbf{p}}) + (\bar{u}_k^{\mathrm{L}} - \bar{p}_k)\nabla\bar{u}_k^{\mathrm{L}} + 2\Omega \times \bar{\mathbf{u}}^{\mathrm{L}} + \nabla\Pi - \bar{T}^{\mathrm{L}}\nabla\bar{s}^{\mathrm{L}} = 0. \tag{1.5}$$

Of course, the GLM equations are also not closed. For closure, they require certain statistical properties of the Lagrangian disturbance ξ to be prescribed. The quantity $\bar{\mathbf{p}}$ in the GLM motion equation (1.5) is the *pseudomomentum vector*, a mean quantity defined by

$$\bar{\mathbf{p}} \equiv -\overline{[u_k^\ell + (\Omega \times \xi)_k]\nabla\xi^k}, \tag{1.6}$$

where $u_k^\ell = \mathrm{D}^{\mathrm{L}}\xi_k/\mathrm{D}t$. The mean potential Π in (1.5) has the form

$$\Pi = \overline{h(p^\xi, s^\xi)} + \bar{\Phi}^{\mathrm{L}}(\mathbf{x}) - \tfrac{1}{2}\overline{\mathbf{u}^\xi \cdot [\mathbf{u}^\xi + 2\Omega \times \xi]}, \tag{1.7}$$

in which the mean specific enthalpy

$$\overline{h(p^\xi, s^\xi)} \equiv \overline{e(D^\xi, s^\xi)} + \overline{(p^\xi/D^\xi)}$$

involves the mean specific internal energy as a function of mass density D^ξ and specific entropy s^ξ. The quantity $\bar{\Phi}^{\mathrm{L}}(\mathbf{x}) = \overline{\Phi^\xi(\mathbf{x})}$ is the Lagrangian mean of an external potential Φ. We note that $s^\xi = \bar{s}^{\mathrm{L}}$ since the specific entropy is a Lagrangian variable in the adiabatic case. The partial derivative $\bar{T}^{\mathrm{L}} = \partial\overline{e(D^\xi, \bar{s}^{\mathrm{L}})}/\partial\bar{s}^{\mathrm{L}}$ is the Lagrangian-mean temperature. For an adiabatic compressible fluid, the thermodynamic first law following a fluid parcel is

$$\mathrm{d}e(D^\xi, \bar{s}^{\mathrm{L}}) = -p^\xi \,\mathrm{d}\left(\frac{1}{D^\xi}\right) + T^\xi \,\mathrm{d}\bar{s}^{\mathrm{L}}.$$

Hence, its Eulerian mean becomes, upon using $\tilde{D} = D^\xi \mathcal{J}$ from mass conservation,

$$\mathrm{d}\overline{e(D^\xi, \bar{s}^{\mathrm{L}})} = -\frac{1}{\tilde{D}}\overline{(p^\xi \mathrm{d}\mathcal{J})} + \frac{1}{\tilde{D}}\overline{(p^\xi/D^\xi)}\,\mathrm{d}\tilde{D} + \bar{T}^{\mathrm{L}}\,\mathrm{d}\bar{s}^{\mathrm{L}}. \tag{1.8}$$

Remark.

- The determinant $\mathcal{J} = \det(\nabla_{\mathbf{x}}(\mathbf{x} + \xi))$ is a fluctuating quantity, not a mean fluid quantity. Therefore, $\mathcal{J}$ will not contribute to variations with respect to mean fluid quantities. However, $\delta\mathcal{J} = K_k^j(\partial\delta\xi^k/\partial x^j)$ with cofactor $K_k^j = \mathcal{J}\partial x^j/\partial(x^k + \xi^k)$ does contribute to variations with respect to ξ in the self-consistent wave, mean flow interaction (WMFI) theory of Gjaja and Holm [7]. Such variations also arise, e.g., in the Lagrangian stability analysis of the equilibrium solutions of the GLM equations. See, e.g. [3] for a discussion of Hamilton's principle for the Lagrangian disturbance ξ and its relation to the wave action density of the WMFI theory.
- Thus, the transformation properties of the GLM theory provide proper definitions of the *thermodynamic derivatives of mean constitutive relations* with respect to GLM average fluid variables.

1.2. Pseudomomentum and the transport structure of the GLM motion equation

The significance of the pseudomomentum $\mathbf{p}$ to the transport structure of the GLM equations can be understood from the Lagrangian mean of the contour integral appearing in Kelvin's circulation theorem for fluid motion in a rotating frame. The rotation frequency Ω is allowed to depend on position and is given by $2\Omega(\mathbf{x}^\xi) = (\mathrm{curl}\,\mathbf{R})^\xi$. The rotation potential $\mathbf{R}(\mathbf{x}^\xi)$ is decomposed in standard GLM fashion as $\mathbf{R}^\xi = \bar{\mathbf{R}}^{\mathrm{L}} + \mathbf{R}^\ell$. (A constant rotation frequency is recovered from specializing to $\mathbf{R}^\xi = \Omega \times \mathbf{x}^\xi$.)

The GLM average of Kelvin's circulation integral is defined as

$$\begin{aligned}\overline{I(t)} &= \overline{\oint_{\gamma^\xi(t)} (\mathbf{u}^\xi + \mathbf{R}(\mathbf{x}^\xi)) \cdot \mathrm{d}\mathbf{x}^\xi} = \overline{\oint_{\gamma^\xi(t)} (\bar{\mathbf{u}}^{\mathrm{L}} + \bar{\mathbf{R}}^{\mathrm{L}} + \mathbf{u}^\ell + \mathbf{R}^\ell) \cdot (\mathrm{d}\mathbf{x} + \mathrm{d}\xi)} \\ &= \oint_{\bar{\gamma}^{\mathrm{L}}(t)} (\bar{\mathbf{u}}^{\mathrm{L}} + \bar{\mathbf{R}}^{\mathrm{L}} + \overline{[u_k^\ell + R_k^\ell]\nabla\xi^k}) \cdot \mathrm{d}\mathbf{x} = \oint_{\bar{\gamma}^{\mathrm{L}}(t)} (\bar{\mathbf{u}}^{\mathrm{L}} + \bar{\mathbf{R}}^{\mathrm{L}} - \mathbf{p}) \cdot \mathrm{d}\mathbf{x},\end{aligned}$$

where the contour $\bar{\gamma}^{\mathrm{L}}(t)$ moves with velocity $\bar{\mathbf{u}}^{\mathrm{L}}$, since it follows the fluid parcels as the average is taken. Thus, the Lagrangian mean leaves invariant the *form* of the Kelvin integral, while averaging the *velocity* of its contour. In addition, the pseudomomentum vector $\mathbf{p}$ defined in (1.6) appears in the GLM averaged Kelvin integral as the Lagrangian-mean contribution of the fluctuations to the GLM averaged *integrand*.

The time derivative of the GLM averaged Kelvin circulation integral is

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{I(t)} = \oint_{\bar{\gamma}^{\mathrm{L}}(t)} [(\partial_t + \bar{\mathbf{u}}^{\mathrm{L}}\cdot\nabla)(\bar{\mathbf{u}}^{\mathrm{L}} - \mathbf{p}) + (\bar{u}_k^{\mathrm{L}} - p_k)\nabla\bar{u}^{\mathrm{L}k} + 2\Omega\times\bar{\mathbf{u}}^{\mathrm{L}}]\cdot\mathrm{d}\mathbf{x}.$$

The combination of terms in the integrand defines the *transport structure* of the GLM theory. From the GLM motion equation (1.5), one now finds the GLM Kelvin circulation theorem for adiabatic compressible flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{I(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\oint_{c(\bar{\mathbf{u}}^{\mathrm{L}})} (\bar{\mathbf{u}}^{\mathrm{L}} + \bar{\mathbf{R}}^{\mathrm{L}} - \mathbf{p})\cdot\mathrm{d}\mathbf{x} = \oint_{c(\bar{\mathbf{u}}^{\mathrm{L}})} \bar{T}^{\mathrm{L}}\,\mathrm{d}\bar{s}^{\mathrm{L}}.$$

Thus, the Lagrangian mean *averages the velocity* of the fluid parcels on the Kelvin circulation loop, while it *adds the mean contribution* of the fluctuations to the Kelvin circulation integrand. In particular, upon taking the Lagrangian mean, the velocity of fluid parcels on the circulation loop and the velocity appearing in the circulation integrand are *different*.

In the isentropic case (or, if the loop $c(\bar{\mathbf{u}}^{\mathrm{L}})$ moving with the Lagrangian-mean flow lies entirely on a level surface of $\bar{s}^{\mathrm{L}}$) then the right-hand side vanishes, and one finds the "generalized Charney-Drazin theorem" for transient waves discussed in [2].

2. EP formulation of the GLM equations using an averaged variational principle

2.1. EP Averaging Result for GLM equations

Most of the important properties of the GLM equations are discussed in [2]. Many of these properties arise from general mathematical structures that are shared by all exact nonlinear ideal fluid theories. With the help of the thermodynamic identity (1.8) for $d\overline{e(D^\xi, \bar{s}^{\mathrm{L}})}$, we shall recast the GLM fluid motion equation (1.5) as an *EP equation*,

$$\frac{\partial}{\partial t}\frac{\delta\bar{\ell}}{\delta\bar{u}_i^{\mathrm{L}}} + \frac{\partial}{\partial x_k}\left(\frac{\delta\bar{\ell}}{\delta\bar{u}_i^{\mathrm{L}}}\bar{u}_k^{\mathrm{L}}\right) + \frac{\delta\bar{\ell}}{\delta\bar{u}_k^{\mathrm{L}}}\frac{\partial\bar{u}_k^{\mathrm{L}}}{\partial x_i} = \tilde{D}\frac{\partial}{\partial x_i}\frac{\delta\bar{\ell}}{\delta\tilde{D}} - \frac{\delta\bar{\ell}}{\delta\bar{s}^{\mathrm{L}}}\frac{\partial\bar{s}^{\mathrm{L}}}{\partial x_i}, \tag{2.1}$$

expressed in terms of variational derivatives of an averaged Lagrangian, $\bar{\ell}(\bar{\mathbf{u}}^{\mathrm{L}}, \tilde{D}, \bar{s}^{\mathrm{L}})$. See [25,26] for an exposition of the mathematical structures that arise in the EP theory of ideal fluids that possess advected quantities such as heat and mass. For GLM, the Eulerian expression of the averaged Lagrangian is

$$\bar{\ell}(\bar{\mathbf{u}}^{\mathrm{L}}, \tilde{D}, \bar{s}^{\mathrm{L}}) = \int \mathrm{d}^3x\,\tilde{D}\left[\frac{1}{2}\overline{\left|\bar{\mathbf{u}}^{\mathrm{L}} + \frac{\mathrm{D}^{\mathrm{L}}\xi}{\mathrm{D}t}\right|^2} + \overline{(\Omega\times\mathbf{x}^\xi)\cdot\left(\bar{\mathbf{u}}^{\mathrm{L}} + \frac{\mathrm{D}^{\mathrm{L}}\xi}{\mathrm{D}t}\right)} - \overline{\Phi(\mathbf{x}^\xi)} - \overline{e(D^\xi, \bar{s}^{\mathrm{L}})}\right]. \tag{2.2}$$

The mean Lagrangian $\bar{\ell} \equiv \int \bar{\mathcal{L}}(\bar{\mathbf{u}}^{\mathrm{L}}, \tilde{D}, \bar{s}^{\mathrm{L}}; \xi)\,\mathrm{d}^3x$ is a straight transcription of the standard Lagrangian for adiabatic fluids into the GLM formalism, followed by taking the Eulerian mean. If desired, the rotation frequency can be allowed to depend on position by replacing $(\Omega\times\mathbf{x}^\xi) \to \mathbf{R}(\mathbf{x}^\xi)$, in which case $2\Omega \to (\mathrm{curl}\,\mathbf{R})^\xi$. The variational derivatives of $\bar{\ell}$ are given by

$$\delta\bar{\ell} = \int \mathrm{d}^3x \left[\tilde{D}(\bar{\mathbf{u}}^{\mathrm{L}} - \bar{\mathbf{p}} + \Omega \times \mathbf{x}) \cdot \delta\bar{\mathbf{u}}^{\mathrm{L}} - \tilde{D}\bar{T}^{\mathrm{L}}\delta\bar{s}^{\mathrm{L}} - \Pi\delta\tilde{D} \right. \\ \left. + \tilde{D}\overline{[u_k^{\ell} + (\Omega \times \xi)_k](\partial_t\delta\xi^k + \bar{\mathbf{u}}^{\mathrm{L}} \cdot \nabla\delta\xi^k)} + \overline{p^{\xi} K_k^j \left(\frac{\partial\delta\xi^k}{\partial x^j}\right)}\right], \tag{2.3}$$

where $K_k^j = \mathcal{J}\partial x^j/\partial(x^k + \xi^k)$ is the cofactor that arises in the thermodynamic identity (1.8). Thus, the pseudomomentum $\bar{\mathbf{p}}$ defined in (1.6), the Lagrangian-mean temperature $\bar{T}^{\mathrm{L}} = \partial\overline{e(D^{\xi}, \bar{s}^{\mathrm{L}})}/\partial\bar{s}^{\mathrm{L}}$ and the potential Π in (1.7) all arise naturally in the variational derivatives of the Lagrangian $\bar{\ell}$ in (2.2) with respect to the mean fluid quantities.

One may verify that the GLM motion equation (1.5) for the mean fluid motion is now recovered by substituting the variational derivatives of $\bar{\ell}$ in $\bar{\mathbf{u}}^{\mathrm{L}}$, $\tilde{D}$ and $\bar{s}^{\mathrm{L}}$ into the EP equation (2.1). This computation places the GLM theory into the EP framework for the averaged Lagrangian (2.2) and, thus, directly proves the following.

Lemma 1 (GLM adiabatic fluids satisfy EP equations). *The GLM motion equation (1.5) for a compressible adiabatic fluid results when variations of the averaged Lagrangian (2.2) are substituted into the EP equation (2.1).*

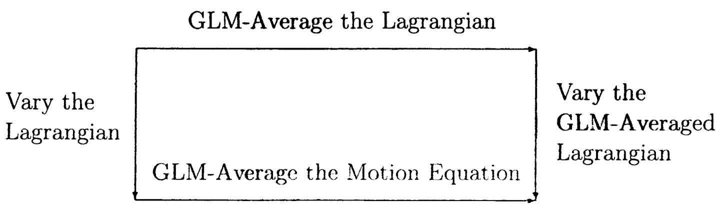
This lemma suggests that a much broader principle is operating, namely the followimg theorem.

Theorem 2.1 (EP Averaging Result). *GLM averaging preserves the four equivalence relations of the EP theorem of Holm et al.* [25,26].

This is verified by the LAEP theorem stated and proved in [23]. Hence, we have the variational property.

Corollary 1 (Variational Reduction Property). *GLM motion equations follow from GLM-averaged EP variational principles.*

The variational reduction property is summarized by the following commutative diagram.



2.1.1. Sketches of Proofs

The EP Averaging Result 2.1 follows from the LAEP theorem of Holm [23]. It also follows "by bare hands" upon using the definition of GLM averaging. First, one observes that the GLM average of a right-invariant Lagrangian is still right-invariant, so the Lagrange-to-Euler reduction and GLM averaging are compatible in the EP theorem of Holm et al. [25,26], which requires this right-invariance. (This observation takes us along the top and down the right side of the commutative diagram above.) Second, the GLM average of the motion equation preserves the transport structure of the Kelvin circulation theorem, which is also implied by the EP theorem. (This is verified by the GLM Kelvin circulation loop analysis in Section 1.2.) Third, by Eq. (1.2), the GLM average preserves the form of the advection relations. Finally, to identify the averages of thermodynamic derivatives that appear in the averaged motion equation and, thus, complete the proof, one uses commutation of exterior derivatives and GLM averaging in the definitions of these average thermodynamic variables. For example, the average temperature of a fluid parcel is correctly defined from the GLM average of the First Law, since thermodynamic relations are applied in ideal fluid theories for each fluid parcel as a *closed system*. Moreover, the *same* definitions are used in the variational derivatives of the averaged Lagrangian. These observations are sufficient to directly prove Result 2.1—the EP Averaging Result. For more details, see [22,23]. There, the EP Averaging Result is visualized as the front face of a cube of six interlocking commutative diagrams representing the equivalence relations of the LAEP Theorem.

Corollary 1 follows immediately from Result 2.1 for any Euler fluid equation that is also an EP equation before the averaging is applied. Moreover, Corollary 1 can also be proven independently by using the Clebsch procedure, e.g., which places the GLM averaged equations and their average advection relations directly into the EP variational framework. Descriptions of the Clebsch procedure in this context are given in [33,24].

The EP Averaging Result and its corollary the variational reduction property allow extension of the exact nonlinear GLM theory to include, e.g., the continuum theory applications of the EP theorem considered in [25,26], and the geophysical fluids applications considered in [1,27]. The remainder of this paper will be devoted to exploring some of the applications of the EP Averaging Result in the small disturbance approximation.

2.2. GLM results arising in the EP framework

The EP framework instills several fundamental properties, including some that the GLM theory is already known to possess. These known properties include the Kelvin circulation theorem, the balance laws for energy and momentum, and the potential vorticity conservation law for GLM. These properties are briefly expressed in the EP framework, as follows. For more details and the original development of the EP theory with advected parameters, see [25,26].

2.2.1. EP Kelvin circulation theorem for adiabatic GLM

The EP motion equation (2.1) can be rewritten in *Lie-derivative form*, as

$$\left(\frac{\partial}{\partial t}+\pounds_{\bar{\mathbf{u}}^{\mathrm{L}}}\right)\left(\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\mathbf{u}^{\mathrm{L}}}\cdot\mathrm{d}\mathbf{x}\right)=d\frac{\delta\bar{\ell}}{\delta\tilde{D}}-\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{s}^{\mathrm{L}}}\,\mathrm{d}\bar{s}^{\mathrm{L}},$$

where $\pounds_{\bar{\mathbf{u}}^{\mathrm{L}}}$ is the Lie derivative with respect to the Lagrangian-mean velocity, $\bar{\mathbf{u}}^{\mathrm{L}}$. Integrating this form of the EP motion equation around a loop $c(\bar{\mathbf{u}}^{\mathrm{L}})$ moving with the average motion of the fluid provides the *Kelvin–Noether theorem* in the EP framework for adiabatic compressible fluids, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c(\bar{\mathbf{u}}^{\mathrm{L}})}\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^{\mathrm{L}}}\cdot\mathrm{d}\mathbf{x}=-\oint_{c(\bar{\mathbf{u}}^{\mathrm{L}})}\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{s}^{\mathrm{L}}}\,\mathrm{d}\bar{s}^{\mathrm{L}}.$$

Hence, for adiabatic compressible GLM flow, from Eq. (2.3) for the required variational derivatives one recovers Eq. (1.2) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c(\bar{\mathbf{u}}^{\mathrm{L}})}(\bar{\mathbf{u}}^{\mathrm{L}}-\bar{\mathbf{p}}+\Omega\times\mathbf{x})\cdot\mathrm{d}\mathbf{x}=\oint_{c(\bar{\mathbf{u}}^{\mathrm{L}})}\bar{T}^{\mathrm{L}}\,\mathrm{d}\bar{s}^{\mathrm{L}}.$$

2.2.2. Energy balance for adiabatic GLM

Legendre transforming the mean Lagrangian $\bar{\ell}$ in Eq. (2.2) yields

$$\bar{E}=\int\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^{\mathrm{L}}}\cdot\bar{\mathbf{u}}^{\mathrm{L}}\,\mathrm{d}^3x-\bar{\ell}=\int\tilde{D}\left[\frac{1}{2}|\bar{\mathbf{u}}^{\mathrm{L}}|^2+\frac{1}{2}\overline{|\mathbf{u}^{\ell}|^2}+\bar{\Phi}^{\mathrm{L}}(\mathbf{x})+\overline{e(D^{\xi},\bar{s}^{\mathrm{L}})}-\overline{(\mathbf{u}^{\ell}+\Omega\times\xi)\cdot\partial_t\xi}\right]\mathrm{d}^3x.$$

Except for the last term, this is the total mean energy of the adiabatic GLM theory. The last term in the energy quantity $\bar{E}$ involves the "pseudoenergy"

$$\bar{e} \equiv \overline{[u_k^\ell + (\Omega \times \xi)_k]\partial_t \xi^k}. \tag{2.4}$$

This term is independent of the internal energy and has a common factor with the pseudomomentum defined earlier

$$-\bar{\mathbf{p}} \equiv \overline{[u_k^\ell + (\Omega \times \xi)_k]\nabla \xi^k}.$$

In fact, these two quantities may be expressed equivalently as

$$\tilde{D}\bar{e} = \overline{\pi_k \partial_t \xi^k} \quad \text{and} \quad \tilde{D}\bar{\mathbf{p}} = -\overline{\pi_k \nabla \xi^k},$$

where $\pi_k \equiv \delta\ell/\delta(\partial_t \xi^k) = \tilde{D}[u_k^\ell + (\Omega \times \xi)_k]$ is the momentum density canonically conjugate to ξ^k, *before* the Eulerian mean is taken in the Lagrangian $\bar{\ell}$.

The spatially integrated pseudoenergy is given by

$$\langle \bar{e} \rangle = \int \tilde{D}\bar{e}\, \mathrm{d}^3x = \int \overline{\pi_k \partial_t \xi^k}\, \mathrm{d}^3x.$$

This term would have *cancelled*, had we performed the complete Legendre transformation,

$$\bar{\mathcal{E}} = \int \left(\frac{\delta \bar{\ell}}{\delta \bar{\mathbf{u}}^{\mathrm{L}}} \cdot \bar{\mathbf{u}}^{\mathrm{L}} + \overline{\frac{\delta \bar{\ell}}{\delta(\partial_t \xi)} \partial_t \xi} \right) \mathrm{d}^3x - \bar{\ell} = \bar{E} + \int \overline{\pi \cdot \partial_t \xi}\, \mathrm{d}^3x$$

in both fluid and wave properties. The pseudoenergy $\bar{e}$ in Eq. (2.4) is thus understood to be the mean mechanical action per unit mass of the fluctuating Lagrangian displacement field. The complete Legendre transformation yields the expected result for the conserved total mean energy for a self-consistent theory

$$\bar{\mathcal{E}} = \int \tilde{D} \left[\frac{1}{2}|\bar{\mathbf{u}}^{\mathrm{L}}|^2 + \frac{1}{2}\overline{|\mathbf{u}^\ell|^2} + \bar{\Phi}^{\mathrm{L}}(\mathbf{x}) + \overline{e(D^\xi, \bar{s}^{\mathrm{L}})} \right] \mathrm{d}^3x.$$

Hence, we find that $\mathrm{d}\bar{E}/\mathrm{d}t = -(\mathrm{d}/\mathrm{d}t)\int \overline{\pi \cdot \partial_t \xi}\, \mathrm{d}^3x = -(\mathrm{d}/\mathrm{d}t)\int \tilde{D}\bar{e}\, \mathrm{d}^3x$, since the total mean energy $\bar{\mathcal{E}}$ must be conserved for a theory with no sources or sinks of energy. This holds by Noether's theorem for the mean Lagrangian $\bar{\ell}$ in Eq. (2.2).

2.2.3. Remark about averaging and conservation laws

Before averaging, the integrated *instantaneous* pseudomomentum is defined as

$$\langle \mathbf{p} \rangle = \int \tilde{D}\mathbf{p}\, \mathrm{d}^3x = -\int \pi_k \nabla \xi^k\, \mathrm{d}^3x.$$

The spatially integrated quantity $\langle \mathbf{p} \rangle$ generates infinitesimal Eulerian spatial shifts of the wave properties as *canonical transformations*. That is,

$$\{\langle \mathbf{p} \rangle, \xi\} = \nabla \xi \quad \text{and} \quad \{\langle \mathbf{p} \rangle, \pi\} = \nabla \pi,$$

where $\{F, H\}$ is the canonical Poisson bracket with $\{\xi(\mathbf{x}'), \pi(\mathbf{x})\} = \delta(\mathbf{x} - \mathbf{x}')$.

Under this canonical Poisson bracket, one may verify the formulas in [20]

$$\mathsf{A} = -\int \pi \cdot \partial_a \xi\, \mathrm{d}^3x \Rightarrow \{\mathsf{A}, \xi\} = \partial_a \xi \quad \text{and} \quad \{\mathsf{A}, \pi\} = \partial_a \pi.$$

That is, the functional A generates a translation in phase space for any parameter a that admits integration by parts. If the solutions in phase space (π, ξ) are averaged over such a parameter, then the averaged generator of the translations, $\bar{\mathsf{A}} = -\int \overline{\pi \cdot \partial_a \xi}\, \mathrm{d}^3x$, will be conserved. For example, the ith component $\bar{p}_i$ of the pseudomomentum would be conserved, if the solutions (π, ξ) were averaged over space in the ith direction.

2.2.4. Remark—the relation between GLM and WMFI

The GLM and WMFI theories are closely related. For example, the WMFI wave action density has the same character as the GLM quantities, pseudomomentum and pseudoenergy, which may also be aptly expressed in terms of a single-frequency WKB wave packet. By varying the wave properties ξ in the averaged Lagrangian as well as the mean fluid properties, Gjaja and Holm [18] constructed a *self-consistent* Lagrangian-mean WMFI theory. This WMFI theory reduced to GLM theory when the statistics of ξ were *prescribed*.

To explain how the wave action density of the WMFI theory is related to the GLM pseudomomentum, we make the following pre-canonical transformation

$$\tilde{D}\bar{\mathbf{p}} \cdot \mathrm{d}\mathbf{x} = -\overline{\pi_k \cdot \nabla \xi^k} \cdot \mathrm{d}\mathbf{x} = -\overline{\pi \cdot \mathrm{d}\xi}.$$

If ξ and π depend on a phase parameter ϕ, we may write the phase-averaged differential relation as

$$-\overline{\pi \cdot \mathrm{d}\xi} = -\overline{\pi_k \partial_\phi \xi^k}\, \mathrm{d}\phi = N\, \mathrm{d}\phi = N\mathbf{k} \cdot \mathrm{d}\mathbf{x},$$

where the wavevector $\mathbf{k}$ is defined by $\mathrm{d}\phi = \nabla\phi \cdot \mathrm{d}\mathbf{x} = \mathbf{k} \cdot \mathrm{d}\mathbf{x}$. Thus, we obtain the wave action density $N = -\overline{\pi_k \partial_\phi \xi^k}$, which is related to the GLM pseudomomentum by $\tilde{D}\bar{\mathbf{p}} = N\mathbf{k}$. For the WKB wavepacket $\xi = \frac{1}{2}(\mathbf{a}\, \mathrm{e}^{\mathrm{i}\phi/\epsilon} + \mathbf{a}^*\, \mathrm{e}^{-\mathrm{i}\phi/\epsilon})$, one finds the formula

$$\frac{N}{\tilde{D}} = -\overline{\left[\frac{\mathrm{D}^{\mathrm{L}}\xi}{\mathrm{D}t} + (\Omega \times \xi)\right] \cdot \partial_\phi \xi} = 2\tilde{\omega}|\mathbf{a}|^2 + 2\mathrm{i}\Omega \cdot \mathbf{a} \times \mathbf{a}^* + 2\left(\mathbf{a} \cdot \frac{\mathrm{D}^{\mathrm{L}}\mathbf{a}^*}{\mathrm{D}t}\right),$$

in which $\tilde{\omega} = -\mathrm{D}^{\mathrm{L}}\phi/\mathrm{D}t = \omega - \mathbf{k} \cdot \bar{\mathbf{u}}^{\mathrm{L}}$ is the Doppler-shifted wave frequency. This formula agrees with the wave action density N appearing in WMFI studies such as that of Gjaja and Holm [18]. As a result of the symmetry under translations in ϕ introduced by phase-averaging the Lagrangian, we have

$$0 = -\frac{\partial}{\partial t}\frac{\partial \bar{\mathcal{L}}}{\partial(\partial_t \phi)} - \mathrm{div}\frac{\partial \bar{\mathcal{L}}}{\partial(\nabla\phi)} = \frac{\partial N}{\partial t} + \frac{\partial}{\partial x^j}(N\bar{u}^{Lj} - \overline{p^\xi K_i^j \partial_\phi \xi^i}),$$

upon using the variational derivatives in Eq. (2.3). Andrews and McIntyre [3] obtain the same conservation law by directly manipulating the GLM motion equation (1.5). This equivalence, of course, is guaranteed by the EP Averaging Result 2.1.

We recover this conservation law as a result of *Noether's theorem* for the averaged Lagrangian. Thus, we have the following the lemma.

Lemma 2. *When averaging introduces an ignorable coordinate, the average of the corresponding canonically conjugate momentum is conserved. In this case, the conserved wave action density N is the phase-averaged generator of phase shifts*.

Remark.

- The GLM pseudoenergy $\bar{e}$ is related to the wave action density N by $\tilde{D}\bar{e} = N\omega$, which again identifies $\bar{e}$ as an action variable.
- The self-consistent WMFI theory is closed by writing the pseudomomentum as $\tilde{D}\bar{\mathbf{p}} = N\mathbf{k}$ and using the *conservation of waves* relation, $\partial_t \mathbf{k} = \nabla\omega$. In this equation, the frequency variable ω must still be determined. Until this point, no small-amplitude assumption has been made. Introducing a small-amplitude approximation

allows the frequency ω to be determined from its dispersion relation in terms of fluid and wave mean properties, as was pioneered in [37,38]. See [18] for more details, including the Lie–Poisson Hamiltonian structure of the self-consistent WMFI theory, which is reminiscent of the Landau two-fluid model of superfluid Helium.

2.2.5. EP momentum balance for adiabatic GLM

The momentum conservation law for the EP theory is (by Noether's theorem)

$$\partial_t \bar{m}_i^{\mathrm{L}} + \partial_j \bar{T}_i^j = \left.\frac{\partial \bar{\mathcal{L}}}{\partial x^i}\right|_{\exp}, \tag{2.5}$$

where $\bar{\mathbf{m}}^{\mathrm{L}} = \delta\bar{\ell}/\delta\bar{\mathbf{u}}^{\mathrm{L}}$ is the Lagrangian-mean momentum density, the stress tensor $\bar{T}_i^j$ is given by

$$\bar{T}_i^j = \bar{m}_i \bar{u}^{\mathrm{L}j} + \delta_i^j \left(\bar{\mathcal{L}} - \tilde{D}\frac{\partial \bar{\mathcal{L}}}{\partial \tilde{D}} \right)$$

and $\partial\bar{\mathcal{L}}/\partial x^i|_{\exp}$ denotes the derivative with respect to the explicit spatial dependence that arises in the mean Lagrangian $\bar{\ell}$ in (2.2) after averaging over the statistics of ξ. For the adiabatic GLM theory, this stress tensor is given by

$$\bar{T}_i^j = \tilde{D}(\bar{u}_i^{\mathrm{L}} - \bar{p}_i + (\varOmega \times \mathbf{x})_i)\bar{u}^{\mathrm{L}j} + \delta_i^j \tilde{D}\overline{\left(\frac{p^\xi}{D^\xi}\right)}. \tag{2.6}$$

The momentum balance law for adiabatic GLM is specified, only after $\partial\bar{\mathcal{L}}/\partial x^i|_{\exp}$ is specified, by giving the spatial dependence in (2.2) of the wave properties and external potential in the Lagrangian density $\bar{\mathcal{L}}$. This is the requirement for obtaining closure in the GLM theory.

2.2.6. Local EP potential vorticity conservation for adiabatic GLM

Invariance of the Lagrangian under diffeomorphisms (interpreted physically as Lagrangian particle relabeling) implies the local conservation law for EP potential vorticity

$$\frac{\mathrm{D}^{\mathrm{L}}}{\mathrm{D}t}\bar{q}^{\mathrm{L}} = 0, \quad \text{where } \bar{q}^{\mathrm{L}} = \frac{1}{\tilde{D}}\nabla\bar{s}^{\mathrm{L}} \cdot \operatorname{curl}\left(\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^{\mathrm{L}}}\right).$$

For the adiabatic GLM case, the potential vorticity is given explicitly as

$$\bar{q}^{\mathrm{L}} = \frac{1}{\tilde{D}}\nabla\bar{s}^{\mathrm{L}} \cdot \operatorname{curl}(\bar{\mathbf{u}}^{\mathrm{L}} - \bar{\mathbf{p}} + \varOmega \times \mathbf{x}).$$

Note the relation of the potential vorticity to the Kelvin circulation theorem. This is particularly apparent when the Kelvin theorem for adiabatic GLM theory is recast in terms of surface integrals using Stokes theorem, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_A \operatorname{curl}(\bar{\mathbf{u}}^{\mathrm{L}} - \bar{\mathbf{p}} + \varOmega \times \mathbf{x}) \cdot \hat{\boldsymbol{n}}\,\mathrm{d}A = \iint_A \nabla\bar{T}^{\mathrm{L}} \times \nabla\bar{s}^{\mathrm{L}} \cdot \hat{\boldsymbol{n}}\,\mathrm{d}A,$$

where the boundary of the surface A is the fluid loop, $\partial A = c(\mathbf{u}^{\mathrm{L}})$.

2.2.7. GLM helicity

The EP helicity is given by

$$\bar{\Lambda}^{\mathrm{L}} = \int \left(\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^{\mathrm{L}}}\right) \cdot \operatorname{curl}\left(\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^{\mathrm{L}}}\right)\mathrm{d}^3x.$$

The corresponding GLM helicity is not conserved in the adiabatic case, although it is conserved in the GLM theory for the three-dimensional barotropic case. (The same is true, before averaging.)

2.3. EP results for the GLM Boussinesq stratified fluid

The Eulerian expression of the averaged Lagrangian for a Boussinesq stratified fluid is

$$\bar{\ell}(\bar{\mathbf{u}}^{\mathrm{L}}, \tilde{D}, \bar{\theta}^{\mathrm{L}}) = \int \left\{ \tilde{D}\left[\frac{1}{2}\overline{\left|\bar{\mathbf{u}}^{\mathrm{L}} + \frac{\mathrm{D}^{\mathrm{L}}\xi}{\mathrm{D}t}\right|^2} + \overline{(\bar{\mathbf{R}}^{\mathrm{L}} + \mathbf{R}^{\ell})\cdot\left(\bar{\mathbf{u}}^{\mathrm{L}} + \frac{\mathrm{D}^{\mathrm{L}}\xi}{\mathrm{D}t}\right)} - \overline{\Phi(\mathbf{x}^{\xi})} - gz\bar{\theta}^{\mathrm{L}}\right] - \overline{p^{\xi}(\tilde{D} - \mathcal{J})}\right\} \mathrm{d}^3x. \quad (2.7)$$

This mean Lagrangian $\bar{\ell} \equiv \int \bar{\mathcal{L}}(\bar{\mathbf{u}}^{\mathrm{L}}, \tilde{D}, \bar{\theta}^{\mathrm{L}})\, \mathrm{d}^3x$ is a straight GLM decomposition of the standard Lagrangian for Boussinesq stratified fluids, followed by taking the Eulerian mean. The relative buoyancy θ is advected as a scalar in the Boussinesq approximation

$$\partial_t\theta + \mathbf{u}\cdot\nabla\theta = 0,$$

so we have already substituted $\theta^{\xi} = \bar{\theta}^{\mathrm{L}}$. The rotation frequency Ω depends on position and is given by $2\Omega(\mathbf{x}^{\xi}) = (\mathrm{curl}\,\mathbf{R})^{\xi}$. The rotation potential is decomposed in standard GLM fashion as $\mathbf{R}^{\xi} = \bar{\mathbf{R}}^{\mathrm{L}} + \mathbf{R}^{\ell}$. Finally, the pressure p^{ξ} is a Lagrange multiplier that imposes the constraint relation defining the conserved GLM density $\tilde{D} = D^{\xi}\mathcal{J}$, for $D^{\xi} = 1$.

Remark. The kinetic energy is the same here as in Eq. (2.2) for the adiabatic compressible fluid, and the relative buoyancy is perfectly analogous to the entropy per unit mass. Moreover, the pressure constraint is also analogous to internal energy. So, one should expect no substantial difference to occur in passing from the adiabatic GLM case to the Boussinesq GLM equations.

2.3.1. Variational derivatives and EP equation for GLM Boussinesq stratified fluid

The variational derivatives required for the EP equation (2.1)—with entropy $\bar{s}^{\mathrm{L}}$ replaced by buoyancy $\bar{\theta}^{\mathrm{L}}$—are obtained from (ignoring variational derivatives in ξ now and henceforth)

$$\delta\bar{\ell} = \int \mathrm{d}^3x[\tilde{D}(\bar{\mathbf{u}}^{\mathrm{L}} - \bar{\mathbf{p}} + \bar{\mathbf{R}}^{\mathrm{L}})\cdot\delta\bar{\mathbf{u}}^{\mathrm{L}} - \tilde{D}gz\delta\bar{\theta}^{\mathrm{L}} - \Pi^{\mathrm{B}}\delta\tilde{D}]. \quad (2.8)$$

Here, the pseudomomentum is defined by $\bar{\mathbf{p}} = -\overline{(u_j^{\ell} + R_j^{\ell})\nabla\xi^j}$ and the Boussinesq potential Π^{B} is defined by

$$\Pi^{\mathrm{B}} = \pi^{\mathrm{B}} + gz\bar{\theta}^{\mathrm{L}} + \bar{\mathbf{u}}^{\mathrm{L}}\cdot\bar{\mathbf{R}}^{\mathrm{L}},$$

where

$$\pi^{\mathrm{B}} = \bar{p}^{\mathrm{L}} + \bar{\Phi}^{\mathrm{L}}(\mathbf{x}) - \tfrac{1}{2}\overline{\mathbf{u}^{\xi}\cdot(\mathbf{u}^{\xi} + 2\mathbf{R}^{\ell})}.$$

Here $\bar{p}^{\mathrm{L}} = \overline{p^{\xi}}$ is the Lagrangian-mean pressure, cf. Eq. (1.7) for the potential in the adiabatic compressible case. We substitute these variational derivatives into the EP equation (2.1), with the analogous replacement $\bar{s}^{\mathrm{L}} \to \bar{\theta}^{\mathrm{L}}$, to find the following GLM motion equation for a stratified Boussinesq fluid in Cartesian coordinates

$$\left[\frac{\mathrm{D}^{\mathrm{L}}}{\mathrm{D}t}(\bar{\mathbf{u}}^{\mathrm{L}} - \bar{\mathbf{p}}) + (\bar{u}_k^{\mathrm{L}} - \bar{p}_k)\nabla\bar{u}_k^{\mathrm{L}}\right] + 2\Omega\times\bar{\mathbf{u}}^{\mathrm{L}} + \nabla\pi^{\mathrm{B}} + g\bar{\theta}^{\mathrm{L}}\hat{\mathbf{z}} = 0. \quad (2.9)$$

Remark.

- Stratified Boussinesq fluids and adiabatic compressible fluids admit very similar forms of the EP Averaging Result.
- The GLM Boussinesq motion equation (2.9) is very similar to the corresponding adiabatic compressible equation (1.5). However, it is different in an important way from the corresponding Eqs. (8.7a) and (9.1) of Andrews and McIntyre [2], which both contain only $\mathrm{D}^{\mathrm{L}}\bar{\mathbf{u}}^{\mathrm{L}}/\mathrm{D}t$, instead of the combination of four terms in square brackets written here. (This error was not repeated in [6], Eq. (9.4).) Only with the correct combination of these four terms can the Kelvin circulation theorem for GLM be satisfied properly.

2.3.2. EP Kelvin circulation theorem for GLM Boussinesq stratified fluid

The EP framework provides the *Kelvin–Noether theorem* for Boussinesq stratified fluid, in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c(\bar{\mathbf{u}}^{\mathrm{L}})}\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^{\mathrm{L}}}\cdot\mathrm{d}\mathbf{x} = -\oint_{c(\bar{\mathbf{u}}^{\mathrm{L}})}\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\theta}^{\mathrm{L}}}\,\mathrm{d}\bar{\theta}^{\mathrm{L}}.$$

Hence, for the GLM Boussinesq stratified fluid one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c(\bar{\mathbf{u}}^{\mathrm{L}})}(\bar{\mathbf{u}}^{\mathrm{L}}-\bar{\mathbf{p}}+\bar{\mathbf{R}}^{\mathrm{L}}(\mathbf{x}))\cdot\mathrm{d}\mathbf{x} = \oint_{c(\bar{\mathbf{u}}^{\mathrm{L}})} gz\,\mathrm{d}\bar{\theta}^{\mathrm{L}},$$

where $\operatorname{curl}\bar{\mathbf{R}}^{\mathrm{L}}(\mathbf{x}) = 2\Omega(\mathbf{x})$. If the loop $c(\bar{\mathbf{u}}^{\mathrm{L}})$ moving with the Lagrangian-mean flow lies entirely on a level surface of $\bar{\theta}^{\mathrm{L}}$, then the right-hand side vanishes, and one recovers for this case the "generalized Charney-Drazin theorem" for transient Boussinesq internal waves, in analogy to the discussion in [2] for the adiabatic compressible case.

2.3.3. Momentum balance for GLM Boussinesq stratified fluid

For a mean Lagrangian density $\bar{\mathcal{L}}$, the EP theory yields the momentum balance

$$\partial_t\bar{m}_i + \partial_j\bar{T}_i^j = \left.\frac{\partial\bar{\mathcal{L}}}{\partial x^i}\right|_{\mathrm{exp}},$$

with terms defined in analogy with the compressible GLM case.

2.3.4. Local potential vorticity conservation for GLM Boussinesq stratified fluid

Invariance of the Lagrangian under diffeomorphisms (interpreted physically as Lagrangian particle relabeling) implies the local conservation law for EP potential vorticity,

$$\frac{\mathrm{D}^{\mathrm{L}}}{\mathrm{D}t}\bar{q}^{\mathrm{L}} = 0, \quad \text{where } \bar{q}^{\mathrm{L}} = \frac{1}{\tilde{D}}\nabla\bar{\theta}^{\mathrm{L}}\cdot\operatorname{curl}\left(\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^{\mathrm{L}}}\right).$$

For the GLM case, the potential vorticity is given explicitly as

$$\bar{q}^{\mathrm{L}} = \frac{1}{\tilde{D}}\nabla\bar{\theta}^{\mathrm{L}}\cdot\operatorname{curl}(\bar{\mathbf{u}}^{\mathrm{L}}-\bar{\mathbf{p}}+\bar{\mathbf{R}}^{\mathrm{L}}(\mathbf{x})).$$

Again, the EP framework explains the relation of the potential vorticity to the Kelvin circulation theorem.

Other considerations in the EP framework for the GLM Boussinesq stratified fluid closely follow the developments for the GLM adiabatic fluid, modulo simple adjustments for replacing $\bar{s}^{\mathrm{L}} \to \bar{\theta}^{\mathrm{L}}$, in the momentum and energy balance laws, e.g.

2.4. Section summary

This section shows that passing from the Euler equations for ideal compressible and incompressible fluids to the GLM equations admits the *EP Averaging Result*. Namely, under GLM averaging, the EP equations for the averaged Lagrangian are identical to the averaged EP equations. Because of the EP Averaging Result, one finds that the Kelvin circulation theorem, the balances for energy and momentum and the local conservation law for potential vorticity all arise as general features of GLM-averaged EP fluid theories. Concepts in GLM theory such as pseudomomentum and wave action density also arise naturally as general features in the EP context.

Thus, the EP Averaging Result places the exact nonlinear GLM theory into the realm of *averaged Lagrangians* for Eulerian fluid mechanics in the EP framework. This framework allows further structure-preserving approximations of the GLM equations to be made using the EP variational formulation.

Being derivable in the EP framework, the GLM theory also possesses other fundamental structure that is shared by all ideal fluid theories in the EP framework. In particular, the EP framework leads to the Lie–Poisson Hamiltonian formulation for GLM theory, as well as to the potential–vorticity Casimirs associated with this Lie–Poisson bracket. In turn, this structure leads to the energy-Casimir method for characterizing equilibrium solutions of the GLM equations for ideal fluids as critical points of a constrained energy and for establishing their nonlinear Liapunov stability conditions. For an explanation of this additional structure and many applications in fluids and plasmas, see [28].

All of these additional features are now available to the GLM theory of fluid dynamics. However, we shall forego investigating these other implications here and pass to the formulation of an approximate set of Eulerian mean equations based on a small-amplitude approximation of the GLM theory. We refer to [25,26] for detailed descriptions, derivations and basic references to other works concerning the underlying geometry associated with the EP framework for ideal fluids with advected quantities.

Remark.

- The GLM equations may also be obtained by averaging in Hamilton's principle at constant fluid parcel label in the Lagrangian description, then transforming the result to the Eulerian description and again using the EP theory. This approach was taken in [18] in developing a self-consistent WMFI theory for the Boussinesq stratified case. The same approach was taken by Holm [21] in developing nonlinear THC for both compressible and incompressible flows. The equivalence of these other approaches to the present approach is proven by the LAEP theorem in [22,23].
- Regarding stability of the GLM solutions, see [3] for discussion of a variational principle for linear evolution of small disturbances of a Lagrangian-mean flow. In this regard, see also the classical works mentioned earlier on Lagrangian fluid stability analysis and WMFI theory.

3. Linearized Eulerian/Lagrangian fluctuation relations

In principle, the GLM theory is more accurate than an Eulerian mean theory, because its scalar advection relations hold exactly, and it preserves the EP structure of the original unapproximated equations. That is, being an EP theory, GLM preserves the standard ideal fluid relations for energy, momentum and potential vorticity, as well as possessing a Kelvin circulation theorem. However, the results of any Lagrangian-mean theory are difficult to interpret accurately in an Eulerian setting. In addition, the Lagrangian-mean statistics themselves are affected by the mean motion at finite-amplitude Lagrangian displacement and, thus, cannot be taken as prescribed quantities. Therefore, one sees the need for an Eulerian mean counterpart to the GLM theory in the small-amplitude approximation. A theory of this type was recently initiated in [6] in the context of the gravity wave parameterization problem.

In preparation for producing a variational complement to the small-amplitude GLM theory, we shall first discuss the linearized Eulerian/Lagrangian fluctuation relations.

3.1. Taylor series approximations of Eulerian fluctuations at linear order in the Lagrangian displacement ξ

In the GLM theory, the displaced fluid velocity is given by

$$\mathbf{u}(\mathbf{x}+\xi, t) = \bar{\mathbf{u}}^{\mathrm{L}}(\mathbf{x}, t) + \mathbf{u}^{\ell}(\mathbf{x}, t),$$

where

$$\mathbf{u}^{\ell}(\mathbf{x}, t) = \frac{\partial}{\partial t}\xi + \bar{\mathbf{u}}^{\mathrm{L}} \cdot \nabla\xi \equiv \frac{\mathrm{D}^{\mathrm{L}}\xi}{\mathrm{D}t}.$$

A Taylor series approximation shows that the Eulerian velocity fluctuation $\mathbf{u}'$ is related to the Lagrangian disturbance velocity $\mathbf{u}^{\ell}$, as well as the fluctuating displacement ξ and the Eulerian mean velocity $\bar{\mathbf{u}}$ at linear order in ξ by

$$\mathbf{u}^{\ell} = \mathbf{u}' + \xi \cdot \nabla\bar{\mathbf{u}}.$$

Therefore, we find the important relation at linear order,

$$\mathbf{u}'(\mathbf{x}, t) = \frac{\partial}{\partial t}\xi + \bar{\mathbf{u}} \cdot \nabla\xi - \xi \cdot \nabla\bar{\mathbf{u}} \quad (\mathbf{u}' \text{ equation}). \tag{3.1}$$

Likewise, for a scalar quantity χ, we have the linear-order relation, $\chi^{\ell} = \chi' + \xi \cdot \nabla\bar{\chi}$. Consequently, we find

$$\chi' = -\xi \cdot \nabla\bar{\chi} \quad (\chi' \text{ equation}) \tag{3.2}$$

for an *advected scalar* χ (since $\chi^{\ell} = 0$ for an advected scalar). For a conserved density, D, the linear-order Taylor approximation is

$$D^{\ell} = D' + \xi \cdot \nabla\bar{D} = -\bar{D}\,\mathrm{div}\,\xi.$$

Consequently, the Eulerian density fluctuation D' and Eulerian mean density $\bar{D}$ are related to the Lagrangian fluctuating displacement at linear order in ξ for a conserved density D by

$$D' = -\mathrm{div}(\bar{D}\xi) \quad (D' \text{ equation}). \tag{3.3}$$

The $\mathbf{u}'$ and D' equations imply

$$(\bar{D}\mathbf{u}' + D'\bar{\mathbf{u}}) = \partial_t(\bar{D}\xi) - \mathrm{curl}(\bar{\mathbf{u}} \times \bar{D}\xi).$$

Taking the divergence of this relation and using the D' equation then implies the *linearized continuity equation*

$$\partial_t D' = -\mathrm{div}(\bar{D}\mathbf{u}' + D'\bar{\mathbf{u}}). \tag{3.4}$$

Remark.

- The $\mathbf{u}'$, χ' and D' equations are *standard* in Lagrangian stability analysis. See [17] for a historical survey of the use of these linearized fluctuation relations, especially in astrophysics.
- The linearized continuity equation and the projection property $\bar{\bar{\chi}} = \bar{\chi}$ for any quantity χ imply that the Eulerian mean density $\bar{D}$ satisfies the usual *continuity equation*

$$\partial_t\bar{D} + \mathrm{div}\,\bar{D}\bar{\mathbf{u}} = 0 \tag{3.5}$$

  in terms of the Eulerian mean velocity $\bar{\mathbf{u}}$.

- The $\mathbf{u}'$ equation (3.1) may also be expressed in geometrical language in terms of the ad-operator defined on the Lie algebra of vector fields. Namely, as the linear relation

$$\mathbf{u}' = \partial_t \xi + \mathrm{ad}_{\bar{\mathbf{u}}} \xi.$$

In this expression, the ad-operator is defined in terms of the commutator operation for vector fields $[\cdot, \cdot]$ by

$$\mathrm{ad}_{\bar{\mathbf{u}}} \xi = [\bar{\mathbf{u}}, \xi] = \bar{\mathbf{u}} \cdot \nabla \xi - \xi \cdot \nabla \bar{\mathbf{u}} = -\pounds_{\xi} \bar{\mathbf{u}}^{\sharp},$$

where superscript $(\cdot)^{\sharp}$ denotes a vector field.
- In geometrical language, the Eulerian fluctuating component of any advected quantity a is given at linear approximation in ξ by

$$a' = -\pounds_{\xi} \bar{a},$$

where $\bar{a}$ is the Eulerian mean and $\pounds_{\xi}$ denotes the Lie derivative corresponding to the diffeomorphism generated by the vector field ξ, the fluctuating displacement of the Lagrangian trajectory away from its mean position $\mathbf{x}$.
- A strong connection exists between the present approach and the GLM approach to WMFI discussed in [18]. In that paper, attention concentrated on modeling the Lagrangian fluid trajectory displacement fluctuation $\xi(\mathbf{x}, t)$ as a WKB wave packet. Here we shall use the linear relations for D' and $\mathbf{u}'$ to derive Eulerian mean fluid equations that approximate the GLM fluid motion equations at second order in the fluctuation displacement ξ.

4. Deriving $g\ell m$—the order $\mathrm{O}(|\xi|^2)$ GLM equations

In this section, we shall obtain a set of Eulerian-mean equations that approximate the GLM equations at second order in the displacement ξ. Following ideas familiar in Lagrangian fluid stability analysis, we shall derive these approximate equations from a variational principle based on first taking the Eulerian mean of the second-variation of the GLM Lagrangian and then using the EP formulation. Our strategy for developing this order $\mathrm{O}(|\xi|^2)$ approximate Eulerian mean counterpart for GLM is as follows.

We base the structure-preserving approximations of the GLM theory implemented here in the EP framework on a two-step procedure. The first step linearizes the Eulerian/Lagrangian fluctuation relation. (This linearization describes how small fluctuations of a given fluid quantity around its Eulerian mean are related to the fluctuating displacement of a Lagrangian fluid parcel trajectory around its mean position.) The second step substitutes these linearized fluctuation relations into the second-variation of the Lagrangian in Hamilton's principle. The Eulerian mean is then applied. This produces an averaged second-variation Hamilton's principle whose coefficients are Eulerian mean second moments of fluctuating Lagrangian displacements. Finally, variations are taken with respect to Eulerian mean fluid quantities and thereby one obtains the averaged motion equation in the EP framework. Thus, the first step of this procedure is reminiscent of the traditional approach in linear Lagrangian stability analysis for fluids. This traditional approach also invokes the second-variation of a fluid action principle with respect to Lagrangian displacements. However, there is a difference—the procedure here involves fluctuating linear displacements from a mean solution, not deterministic linear displacements from a steady solution as in the traditional stability analysis.

The second-variation Lagrangian contains quadratic terms in ξ, whose coefficients depend on the Eulerian mean fluid quantities. The second step of our procedure begins by taking the Eulerian mean of the second-variation Lagrangian over the quadratic moments of these fluctuating displacements, ξ. We then take variations of this averaged Lagrangian, with respect to the Eulerian mean fluid quantities, and use the EP framework. (In Lagrangian stability analysis, first, one does not average and, second, one takes variations with respect to the Lagrangian displacements, not the steady solutions.) The resulting EP equations express the back-reaction effects on the Eulerian mean motion equation of the fluctuating displacements of the Lagrangian trajectories in terms of their Eulerian second moments. This two-step procedure is performed within the EP framework for right-invariant Lagrangians that are defined on the tangent space of a group. For fluids, this is the group of diffeomorphisms representing the fluid motions, including the Lagrangian fluctuating displacements themselves.

We summarize this two-step procedure in symbols, as follows.

Step 1

- Linearize the fluctuation relations to find Eqs. (3.1)–(3.3),

$$D' = -\mathrm{div}\, \bar{D}\xi,\ \chi' = -\xi \cdot \nabla \bar{\chi}, \qquad \mathbf{u}' = \partial_t \xi + \bar{\mathbf{u}} \cdot \nabla \xi - \xi \cdot \nabla \bar{\mathbf{u}}.$$

- Substitute these relations into the second-variation Lagrangian, to form

$$\ell'' = \int [\partial_t \xi \cdot A \cdot \partial_t \xi + \partial_t \xi \cdot B \cdot \xi + \xi \cdot C \cdot \xi]\, \mathrm{d}^3 x,$$

  where A, B, C are matrix operators involving the mean fluid quantities *and their gradients*, i.e., the set $\{\bar{\mathbf{u}}, \bar{D}, \nabla \bar{\mathbf{u}}, \nabla \bar{D}\}$.

Step 2

- Take the Eulerian mean to form the total mean Lagrangian

$$\bar{\ell} = \bar{\ell}_0 + \tfrac{1}{2}\overline{\ell''}.$$

- Derive the $g\ell m$ motion equation for barotropic compressible fluids by computing the EP equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\bar{D}} \frac{\delta \bar{\ell}}{\delta \bar{\mathbf{u}}} + \frac{1}{\bar{D}} \frac{\delta \bar{\ell}}{\delta \bar{u}^j} \nabla \bar{u}^j = \nabla \frac{\delta \bar{\ell}}{\delta \bar{D}}$$

  for the total mean Lagrangian $\bar{\ell}$ by taking its variations

$$\delta \bar{\ell} = \int \left[\left(\frac{\delta \bar{\ell}}{\delta \bar{\mathbf{u}}} \right) \cdot \delta \bar{\mathbf{u}} + \left(\frac{\delta \bar{\ell}}{\delta \bar{D}} \right) \delta \bar{D} \right] \mathrm{d}^3 x.$$

These variational derivatives involve Eulerian means of quadratic combinations of the Lagrangian fluctuating displacement ξ, and its derivatives $\partial_t \xi$ and $\nabla \xi$. For example, one combination that appears is $\overline{\pi_j \nabla \xi^j}$, where $\pi = \frac{1}{2}\delta \ell''/\delta(\partial_t \xi) = A \cdot \partial_t \xi + B \cdot \xi$ is the momentum canonically conjugate to ξ. These Lagrangian quadratic statistical moments are *unknown parameters* in the $g\ell m$ equations that must be independently specified, or modeled, in closing the equations. Thus, a number of modeling decisions must be made in closing any $g\ell m$ model.

In Section 5, we shall discuss the various modeling parameters required to produce a closed $g\ell m$ model. This will be done in the context of simplifying them and constructing a more manageable class of closed equations—the alpha models—obtained by using closures based on Taylor's hypothesis of frozen-in turbulence.

The equations derived from this two-step procedure—being the small-amplitude approximation of the GLM equations—are called $g\ell m$ equations. They are derived within the EP framework. These new equations describe the dynamics of Eulerian mean fluid quantities influenced by small-amplitude fluctuations. Being EP equations, they still retain the properties that result from particle relabeling symmetry. In particular, the $g\ell m$ equations retain the Kelvin–Noether circulation theorem and its associated local conservation law for potential vorticity.

4.1. *Expanding the Lagrangian $\ell(\mathbf{u}, D)$ in Hamilton's principle for barotropic fluids*

Into the Lagrangian ℓ, we substitute $\mathbf{u} = \bar{\mathbf{u}} + \epsilon\mathbf{u}'$ and $D = \bar{D} + \epsilon D'$, then truncate at quadratic order in $\mathbf{u}'$ and D' to find

$$\ell = \ell_0 + \epsilon\ell' + \frac{\epsilon^2}{2}\ell'', \quad \text{with } {}' = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}.$$

Variations of ℓ are given by

$$\ell' = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \ell(\bar{\mathbf{u}} + \epsilon\mathbf{u}', \bar{D} + \epsilon D') = \left\langle \frac{\delta\ell}{\delta\bar{\mathbf{u}}}, \mathbf{u}' \right\rangle + \left\langle \frac{\delta\ell}{\delta\bar{D}}, D' \right\rangle,$$

where $\langle f, g\rangle = \int fg\,\mathrm{d}^3x$, is the L_2 pairing. The quadratic functional ℓ'' is the second-variation of the Lagrangian ℓ in the basis $\mathbf{u}'$ and D'. That is,

$$\ell'' = \langle(\mathbf{u}', D'), D^2\ell(\bar{\mathbf{u}}, \bar{D}) \cdot (\mathbf{u}', D')\rangle. \tag{4.1}$$

(This is the connection to linear Lagrangian fluid stability theory.) Note that we treat ℓ'' genuinely as a second-variation; so there are no double-prime terms, such as $\mathbf{u}''$.

The *averaged Lagrangian* at second order is then

$$\bar{\ell} = \bar{\ell}_0 + \tfrac{1}{2}\epsilon^2\overline{\ell''}, \quad \text{since } \overline{\ell'} = 0, \quad \text{for } \overline{u'} = 0 = \overline{D'}.$$

Recall that the Eulerian mean satisfies the projection property, $\bar{\bar{\mathbf{u}}} = \bar{\mathbf{u}}$, and it commutes with the spatial gradient, $\overline{\nabla\mathbf{u}} = \nabla\bar{\mathbf{u}}$. On substituting the linearized fluctuation relations for D', χ' and $\mathbf{u}'$ into $\overline{\ell''}$, we find the expected quadratic form

$$\overline{\ell''} = \int [\overline{\partial_t\xi \cdot A \cdot \partial_t\xi} + \overline{\partial_t\xi \cdot B \cdot \xi} + \overline{\xi \cdot C \cdot \xi}]\,\mathrm{d}^3x.$$

The A, B, C in this quadratic form are matrix operators involving the mean fluid quantities *and their gradients*, i.e., the set $\{\bar{\mathbf{u}}, \bar{D}, \nabla\bar{\mathbf{u}}, \nabla\bar{D}\}$. Consequently, after taking variations, the contribution from the mean fluctuation Lagrangian $\overline{\ell''}$ to the mean momentum in the corresponding EP equation will depend on *second-gradients* of the mean fluid quantities. The Lagrangian $\overline{\ell''}$ is a functional of the Eulerian mean quadratic moments of the Lagrangian fluctuation displacements. Consequently, the resulting EP equation will also depend parametrically on the second-order statistics of the Lagrangian fluctuations.

Summary. In the EP framework for the $g\ell m$ equations, we expand the Lagrangian to second order in ξ, take its Eulerian mean, vary it with respect to $\bar{\mathbf{u}}$ and $\bar{D}$, and then model the second-order statistics of ξ that appear in the resulting EP motion equation for $\bar{\mathbf{u}}$.

Our next steps are:

1. Compute the mean momentum of the fluctuations

$$\overline{\mathbf{m}''} = \frac{\delta}{\delta\bar{\mathbf{u}}}\left(\frac{1}{2}\overline{\ell''}\right).$$

2. Write the EP $g\ell m$ equations for total momentum

$$\bar{\mathbf{m}} = \frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}}, \quad \bar{\ell} = \bar{\ell}_0 + (\tfrac{1}{2}\overline{\ell''}).$$

3. Obtain a Kelvin circulation theorem for $g\ell m$ equations from their corresponding EP equations and the Kelvin–Noether theorem for these equations.
4. Derive the $g\ell m$ energy balance by Legendre transforming $\bar{\ell}$, the averaged Lagrangian.
5. Derive the $g\ell m$ stress tensor $\bar{T}_i^j$ in the $g\ell m$ momentum balance law, $\partial_t \bar{m}_i + \partial_j \bar{T}_i^j = \partial\mathcal{L}/\partial x^i|_{\exp}$, including the "fluctuation stresses" by invoking Noether's theorem again.
6. Use the result in Section 5 to interpret the Euler-α model stresses, circulation and momentum in $g\ell m$ terms.

4.2. The $g\ell m$ approximations for a barotropic compressible fluid

We shall now drop any dependence of the fluid internal energy on specific entropy, here and in what follows. Thus, we shall treat only the case of a *barotropic, or isentropic, compressible fluid*. We shall evaluate the necessary variational derivatives of $\bar{\ell}$ with respect to $\bar{\mathbf{u}}$ and $\bar{D}$ by using the linearized relations (definitions) for $\mathbf{u}'$ and D' in terms of the infinitesimal generator $\xi(\mathbf{x}, t)$ in Eqs. (3.1) and (3.3).

4.2.1. Eulerian-mean Lagrangian at order $O(|\xi|^2)$

To second order, the Eulerian-mean of the Lagrangian for a barotropic (isentropic) compressible fluid is given by

$$\bar{\ell}(\bar{\mathbf{u}}, \bar{D}) = \bar{\ell}_0 + \frac{1}{2}\overline{\ell''} = \int \left[\frac{1}{2}\bar{D}|\bar{\mathbf{u}}|^2 - \bar{D}e(\bar{D})\right] \mathrm{d}^3x + \int \left[\frac{1}{2}\bar{D}\overline{|\mathbf{u}'|^2} + \overline{D'\mathbf{u}'}\cdot\bar{\mathbf{u}} - \frac{c^2(\bar{D})}{2\bar{D}}\overline{D'^2}\right] \mathrm{d}^3x. \tag{4.2}$$

For such a fluid, the equation of state defines $c^2(\bar{D})$ via

$$\frac{\partial^2}{\partial \bar{D}^2}(\bar{D}e(\bar{D})) = \frac{\partial}{\partial \bar{D}}h(\bar{D}) = \frac{c^2(\bar{D})}{\bar{D}}.$$

Note, before averaging, the quantity ℓ'' in Eq. (4.1) is simply the second-variation of the Lagrangian $\ell(\mathbf{u}, D)$ with respect to the Eulerian mean velocity and density, evaluated at the mean fluid values, $\bar{\mathbf{u}}$ and $\bar{D}$.

The variational derivatives of the mean fluctuational parts of $\bar{\ell}$ are given by

$$\delta\left(\frac{1}{2}\overline{\ell''}\right) = \int \delta\bar{\mathbf{u}}\cdot[\overline{D'\mathbf{u}'} + \overline{\mathrm{ad}^*_\xi(\bar{D}\mathbf{u}' + D'\bar{\mathbf{u}})}] + \delta\bar{D}\left[\frac{1}{2}\overline{|\mathbf{u}'|^2} + \overline{\xi\cdot\nabla(\mathbf{u}'\cdot\bar{\mathbf{u}})} - \overline{D'^2}\frac{\partial}{\partial\bar{D}}\left(\frac{c^2(\bar{D})}{2\bar{D}}\right) - \overline{\xi\cdot\nabla\left(D'\frac{c^2(\bar{D})}{\bar{D}}\right)}\right] \mathrm{d}^3x. \tag{4.3}$$

In these formulae, recall from (3.1) and (3.3) that $D' = -\mathrm{div}(\bar{D}\xi)$ and

$$\mathbf{u}'(\mathbf{x}, t) = \frac{\partial\xi}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\xi - \xi\cdot\nabla\bar{\mathbf{u}} = \partial_t\xi - \mathrm{ad}_\xi\bar{\mathbf{u}}.$$

Remark.

- *Natural boundary conditions* are $\hat{\mathbf{n}}\cdot\bar{\mathbf{u}} = 0$ and $\hat{\mathbf{n}}\cdot\xi = 0$ on the boundary.
- Recall this is the same ξ as in the GLM theory, so we will be able to make direct comparisons between $g\ell m$ and GLM after assembling the EP equations for the order $O(|\xi|^2)$ approximate theory.
- Note that after substituting the linearized approximations for the fluctuations, the mean Lagrangian and its variational derivatives now also depend on the *gradients* of mean fluid properties.

4.3. The mean fluctuation momentum

We express the mean fluctuational momentum in various computationally useful equivalent forms as

$$\overline{\mathbf{m}''} = \frac{\delta}{\delta\bar{\mathbf{u}}}(\tfrac{1}{2}\overline{\ell''}) = \overline{D'\mathbf{u}'} + \overline{\mathrm{ad}^*_\xi(\bar{D}\mathbf{u}' + D'\bar{\mathbf{u}})} = \bar{D}(\overline{\xi\cdot\nabla\mathbf{u}'} + \overline{u'_j\nabla\xi^j}) + \overline{\mathrm{ad}^*_\xi D'\bar{\mathbf{u}}}$$
$$= \bar{D}\overline{\pounds_\xi(\mathbf{u}')^\flat} + \overline{\mathrm{ad}^*_\xi D'\bar{\mathbf{u}}} \equiv \bar{D}(\bar{\mathbf{u}}^{\mathrm{S}} - \bar{\mathbf{p}}) + \overline{\mathrm{ad}^*_\xi D'\bar{\mathbf{u}}},$$

where superscript 'flat' $(\cdot)^\flat$ denotes a one-form and one defines

$$\bar{\mathbf{u}}^{\mathrm{S}} \equiv \overline{\xi\cdot\nabla\mathbf{u}'} \quad \text{(Stokes mean drift velocity)} \quad \text{and} \quad \bar{\mathbf{p}} \equiv -\overline{u'_j\nabla\xi^j} \quad \text{(GLM pseudomentum)}$$

In Cartesian components, the geometrical combinations $\overline{\pounds_\xi(\mathbf{u}')^\flat}$ and $\overline{\mathrm{ad}^*_\xi D'\bar{\mathbf{u}}}$ are expressed as

$$(\overline{\pounds_\xi(\mathbf{u}')^\flat})_i = \overline{\xi^j u'_{i,j}} + \overline{u'_j\xi^j_{,i}} = (\bar{\mathbf{u}}^{\mathrm{S}} - \bar{\mathbf{p}})_i$$

and

$$(\overline{\mathrm{ad}^*_\xi D'\bar{\mathbf{u}}})_i = \partial_j(\bar{u}_i\overline{D'\xi^j}) + \bar{u}_j\overline{D'\partial_i\xi^j}.$$

These are recurring combinations of terms, reappearing throughout the $g\ell m$ theory.

4.3.1. $g\ell m$ pseudomomentum

Before the second-variation Lagrangian for the $g\ell m$ theory is averaged, one finds the momentum canonically conjugate to ξ, given by, cf. the linearized continuity equation (3.4)

$$\pi = \frac{\delta\bar{\ell}}{\delta(\partial_t\xi)} = (\bar{D}\mathbf{u}' + D'\bar{\mathbf{u}}).$$

The corresponding pseudomomentum for the $g\ell m$ theory is then given by

$$\tilde{\mathbf{p}} = -\overline{\pi_k\nabla\xi^k} = -\overline{(\bar{D}u'_k + D'\bar{u}_k)\nabla\xi^k}.$$

Thus, our earlier discussion indicates that a WMFI version of compressible $g\ell m$ theory would possess a conserved wave action density given by

$$N = -\overline{\pi_k\partial_\phi\xi^k} = -\overline{(\bar{D}u'_k + D'\bar{u}_k)\partial_\phi\xi^k}.$$

4.3.2. Mean fluctuation momentum—incompressible case

The mean fluctuation momentum takes a simpler form in the incompressible case. Recall $D' = -\mathrm{div}(\bar{D}\xi)$. Consequently, $\mathrm{div}\,\mathbf{u} = 0$ (which implies $\mathrm{div}\,\bar{\mathbf{u}} = 0 = \mathrm{div}\,\mathbf{u}'$) is consistent with setting $\bar{D} = 1$ in the mean continuity equation

$$\partial_t\bar{D} = -\mathrm{div}(\bar{D}\bar{\mathbf{u}}).$$

Also setting $\bar{D} = 1$ in the density fluctuation gives

$$D'|_{\bar{D}=1} = -\mathrm{div}\,\xi.$$

Taking the divergence of the $\mathbf{u}'$ equation (3.1) then yields

$$\mathrm{div}\,\mathbf{u}' = 0 = \partial_t(\mathrm{div}\,\xi) + \bar{\mathbf{u}}\cdot\nabla(\mathrm{div}\xi).$$

So div $\xi = 0$ is preserved, which means we may choose *initial conditions* so that $D' = 0$. Thus, D' vanishes (after taking variations) in the incompressible case, upon invoking the preserved initial conditions $\bar{D} = 1$ and div $\xi = 0$.

Upon setting $D' = 0$ in the formulas for the incompressible case, the mean momentum may be expressed equivalently as

$$\bar{\mathbf{m}} = \frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}}\bigg|_{\bar{D}=1} = \bar{\mathbf{u}} + \overline{\mathrm{ad}^*_\xi \mathbf{u}'} = \bar{\mathbf{u}} + \overline{\pounds_\xi (\mathbf{u}')^\flat} = \bar{\mathbf{u}} + \overline{\xi \cdot \nabla \mathbf{u}'} + \overline{u'_j \nabla \xi^j} = \bar{\mathbf{u}} + \bar{\mathbf{u}}^{\mathrm{S}} - \bar{\mathbf{p}}$$
$$= \bar{\mathbf{u}} - \overline{\xi \times \mathrm{curl}\, \mathbf{u}'} + \nabla \overline{(\xi \cdot \mathbf{u}')}. \tag{4.4}$$

Here the quantities $\bar{\mathbf{u}}^{\mathrm{S}}$ and $\bar{\mathbf{p}}$ are the same as in the GLM theory, when rotation is absent.

4.3.3. *Simplifications in the $g\ell m$ Lagrangian for incompressible mean flow*

For incompressible mean flow, the second-order Eulerian mean $g\ell m$ Lagrangian (4.2) reduces to the remarkably simple form

$$\bar{\ell}(\bar{\mathbf{u}}, \bar{D}) = \int \left[\frac{1}{2}\bar{D}(|\bar{\mathbf{u}}|^2 + \overline{|\mathbf{u}'|^2}) + \bar{p}(1 - \bar{D})\right] \mathrm{d}^3 x. \tag{4.5}$$

Here the pressure constraint enforces $\bar{D} = 1$. As a result, the Eulerian mean velocity $\bar{\mathbf{u}}$ satisfying the usual continuity equation (3.5) is incompressible. Thus, in this case, the fluctuations contribute only to the mean $g\ell m$ kinetic energy in the Lagrangian $\bar{\ell}$ and the Eulerian mean flow is incompressible.

4.4. *$g\ell m$ results arising in the EP framework*

4.4.1. *The motion equation for barotropic $g\ell m$*

For the $g\ell m$ theory in which $\bar{\ell} \equiv \int \bar{\mathcal{L}}(\bar{\mathbf{u}}, \nabla\bar{\mathbf{u}}, \bar{D}, \nabla\bar{D}; \xi(\mathbf{x}, t))\, \mathrm{d}^3 x$, the EP framework yields the equations of motion

$$\partial_t \bar{m}_i + \partial_j(\bar{m}_i \bar{u}^j) + \bar{m}_j \partial_i \bar{u}^j = \bar{D}\frac{\partial}{\partial x^i}\frac{\delta\bar{\ell}}{\delta\bar{D}} \quad \text{and} \quad \partial_t \bar{D} + \mathrm{div}\, \bar{D}\bar{\mathbf{u}} = 0.$$

Here the total mean momentum $\bar{m}_i$ for $g\ell m$ is defined by

$$\bar{m}_i = \frac{\delta\bar{\ell}}{\delta\bar{u}^i} = \frac{\partial\bar{\mathcal{L}}}{\partial\bar{u}^i} - \partial_k \frac{\partial\bar{\mathcal{L}}}{\partial\bar{u}^i_{,k}} = \bar{D}(\bar{u}_i + \bar{u}^{\mathrm{S}}_i - \bar{p}_i) + (\overline{\mathrm{ad}^*_\xi D'\bar{\mathbf{u}}})_i = \bar{D}\bar{u}_i + \overline{D'u'_i} + (\mathrm{ad}^*_\xi(\bar{D}\mathbf{u}' + D'\bar{\mathbf{u}}))_i, \tag{4.6}$$

where $\bar{u}^{\mathrm{S}}_i$ is the GLM Stokes correction and $\bar{p}_i$ is the GLM pseudomomentum given earlier. The variational derivative with respect to mean density is obtained from

$$\frac{\delta\bar{\ell}}{\delta\bar{D}} = \frac{\partial\bar{\mathcal{L}}}{\partial\bar{D}} - \partial_k \frac{\partial\bar{\mathcal{L}}}{\partial\bar{D}_{,k}}.$$

Remark.

- Note, the *linear* fluctuation relations modify the $g\ell m$ total mean momentum $\bar{m}_i$, which, however, appears in the *nonlinearity* of the EP motion equation for the $g\ell m$ theory.
- The combination of *Lagrangian-mean velocity* $\bar{\mathbf{u}}^{\mathrm{L}}$ and pseudomomentum $\bar{\mathbf{p}}$ appearing as $\bar{\mathbf{u}} + \bar{\mathbf{u}}^{\mathrm{S}} - \bar{\mathbf{p}} = \bar{\mathbf{u}}^{\mathrm{L}} - \bar{\mathbf{p}}$ in the total mean momentum for $g\ell m$ also appears in the same way in the GLM theory. Here, however, this combination also adds to $\bar{D}^{-1}\overline{\mathrm{ad}^*_\xi D'\bar{\mathbf{u}}}$.

4.4.2. Mean Kelvin circulation theorem for barotropic $g\ell m$

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c(\bar{\mathbf{u}})}\frac{1}{\bar{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}} = \oint_{c(\bar{\mathbf{u}})}\nabla\frac{\delta\bar{\ell}}{\delta\bar{D}}\cdot\mathrm{d}\mathbf{x} = 0,$$

where for $g\ell m$ one has, in the compressible barotropic case

$$\frac{1}{\bar{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}} = \bar{\mathbf{u}} + \bar{\mathbf{u}}^{\mathrm{S}} - \bar{\mathbf{p}} + \frac{1}{\bar{D}}\overline{\mathrm{ad}^*_{\xi} D'\bar{\mathbf{u}}}.$$

In contrast, for the incompressible case, $\mathrm{div}\,\bar{\mathbf{u}} = 0$ and one again sets $\bar{D} = 1$ and $D' = 0$ in this formula (after taking variations) thereby dropping the last term. Thus, in the incompressible case, the contributions to the circulation integrands are the *same* for both $g\ell m$ and GLM theories. However, the velocities of the fluid loops in the Kelvin circulation theorems are *different*. They are $\bar{\mathbf{u}}$ for $g\ell m$ and $\bar{\mathbf{u}}^{\mathrm{L}}$ for GLM.

Remark.

- When $\mathrm{curl}(\bar{\mathbf{u}}^{\mathrm{S}} - \bar{\mathbf{p}} + \bar{D}^{-1}\overline{\mathrm{ad}^*_{\xi} D'\bar{\mathbf{u}}})$ vanishes, this is the Charney-Drazin "nonacceleration theorem" for $g\ell m$ for barotropic compressible fluids. See [2] for their discussion of the GLM case.
- The *Eulerian mean vorticity* due to the fluctuations in the *incompressible* case is

$$\mathrm{curl}(\bar{\mathbf{u}}^{\mathrm{S}} - \bar{\mathbf{p}}) = -\mathrm{curl}\overline{(\xi\times\omega')} = \overline{\xi\cdot\nabla\omega'} - \overline{\omega'\cdot\nabla\xi} = \overline{\mathrm{ad}_{\xi}\omega'}. \tag{4.7}$$

For potential fluctuations, one sets $\omega' = 0$, where $\omega' = \mathrm{curl}\,\mathbf{u}'$.

4.4.3. Momentum balance for barotropic $g\ell m$

For a mean Lagrangian density $\bar{\mathcal{L}}$, the EP theory yields the momentum balance

$$\partial_t\bar{m}_i + \partial_j\bar{T}_i^j = \left.\frac{\partial\bar{\mathcal{L}}}{\partial x^i}\right|_{\mathrm{exp}},$$

where the total mean momentum $\bar{m}_i$ for barotropic $g\ell m$ is evaluated in Eq. (4.6). The stress tensor is defined in the EP theory for this class of Lagrangians as

$$\bar{T}_i^j = \bar{m}_i\bar{u}^j + \delta_i^j\left(\bar{\mathcal{L}} - \bar{D}\frac{\partial\bar{\mathcal{L}}}{\partial\bar{D}} + \bar{D}\partial_k\frac{\partial\bar{\mathcal{L}}}{\partial\bar{D}_{,k}}\right) - \bar{u}^k_{,i}\frac{\partial\bar{\mathcal{L}}}{\partial\bar{u}^k_{,j}} - \bar{D}_{,i}\frac{\partial\bar{\mathcal{L}}}{\partial\bar{D}_{,j}}. \tag{4.8}$$

Explicitly evaluating the partial derivatives of $\bar{\ell}$ for $g\ell m$ gives

$$\begin{aligned}\bar{T}_i^j &= \bar{m}_i\bar{u}^j + \delta_i^j\left[p(\bar{D}) + \overline{D'\mathbf{u}'}\cdot\bar{\mathbf{u}} - \frac{c^2(\bar{D})}{\bar{D}}\overline{D'^2}\right]\\ &\quad + \bar{D}\delta_i^j\left[-\overline{\xi\cdot\nabla(\mathbf{u}'\cdot\bar{\mathbf{u}})} + \overline{D'^2}\frac{\partial}{\partial\bar{D}}\left(\frac{c^2(\bar{D})}{2\bar{D}}\right) + \overline{\xi\cdot\nabla\left(D'\frac{c^2(\bar{D})}{\bar{D}}\right)}\right]\\ &\quad + \bar{u}^k_{,i}\overline{(D'\bar{u}_k + \bar{D}u'_k)\xi^j} + \bar{D}_{,i}\left(\overline{\xi^j\mathbf{u}'}\cdot\bar{\mathbf{u}} - \frac{c^2(\bar{D})}{\bar{D}}\overline{D'\xi^j}\right).\end{aligned} \tag{4.9}$$

As with GLM theory, the momentum balance law is specified, only after $\partial\bar{\mathcal{L}}/\partial x^i|_{\mathrm{exp}}$ is known. This requires specifying the explicit spatial dependence in (4.2) of the wave properties and external potential in the Lagrangian density $\bar{\mathcal{L}}$ for $g\ell m$.

Remark. Thus, the form of the theory is fixed—it is the EP theory. However, its manifestations and channels for expressing energy exchange are many. Even in the barotropic case, e.g., there are many different contributions to the stress tensor from the Lagrangian fluctuations ξ. These contributions are primarily isotropic, including the term $\partial\bar{\mathcal{L}}/\partial x^i|_{\exp}$.

4.4.4. Energy balance for barotropic $g\ell m$

A Legendre transformation gives the energy quantity for the $g\ell m$ fluid flow, namely,

$$\bar{E} = \left\langle \frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}} \cdot \bar{\mathbf{u}} \right\rangle - \bar{\ell}(\bar{\mathbf{u}}, \bar{D}, \xi) = \int \left[\frac{1}{2}\bar{D}|\bar{\mathbf{u}}|^2 + \bar{D}e(\bar{D}) + \frac{1}{2}\bar{D}\overline{|\mathbf{u}'|^2} + \overline{D'\mathbf{u}'} \cdot \bar{\mathbf{u}} + \frac{c^2(\bar{D})}{2\bar{D}}\overline{D'^2} \right] \mathrm{d}^3x$$
$$- \int (\bar{D}\overline{|\mathbf{u}'|^2} + \overline{D'\mathbf{u}'} \cdot \bar{\mathbf{u}} - \overline{(\mathrm{ad}^*_\xi(\bar{D}\mathbf{u}' + D'\bar{\mathbf{u}}))} \cdot \bar{\mathbf{u}})\, \mathrm{d}^3x \equiv \left[E(\bar{\mathbf{u}}, \bar{D}) + \frac{1}{2}\overline{E''} \right] - \int \overline{\left(\frac{\delta\bar{\ell}}{\delta(\partial_t\xi)} \cdot \partial_t\xi \right)}\, \mathrm{d}^3x.$$

We recognize the last integral term as $\int \overline{\pi \cdot \partial_t\xi}\, \mathrm{d}^3x$, the total "pseudoenergy" for the $g\ell m$ theory. Hence, just as for the GLM theory, but now with correspondingly different definitions of terms, we find that $\mathrm{d}\bar{E}/\mathrm{d}t = -(\mathrm{d}/\mathrm{d}t)\int \overline{\pi \cdot \partial_t\xi}\, \mathrm{d}^3x$, in $g\ell m$ theory, since the mean total energy must be conserved for a self-consistently coupled theory arising from Hamilton's principle.

Remark. The quantity $\frac{1}{2}\overline{E''}$ is the same as the approximately conserved expression from acoustics due to [5]:

$$\frac{1}{2}\overline{E''} = \int \left[\frac{1}{2}\bar{D}\overline{|\mathbf{u}'|^2} + \overline{D'\mathbf{u}'} \cdot \bar{\mathbf{u}} + \frac{c^2(\bar{D})}{2\bar{D}}\overline{D'^2} \right] \mathrm{d}^3x$$

as discussed in [3]. However, this quantity is not the pseudoenergy for barotropic $g\ell m$ theory.

4.5. Remarks about $g\ell m$ closure and rapid distortion theory

- The $g\ell m$ theory linearizes the $\mathbf{u}'$ equation, so it neglects the nonlinear term $\mathrm{div}(\mathbf{u}'\mathbf{u}' - \overline{\mathbf{u}'\mathbf{u}'})$ that appears in the $\mathbf{u}'$ equation for Reynolds turbulence closure in the Eulerian mean setting.
- *Rapid distortion theory.* Based on ideas from Lagrangian stability analysis and closely related to ideas from WMFI theory, the $g\ell m$ equations are also related to ideas from rapid distortion theory (RDT). See [7] for an interesting discussion of the close connections between RDT and WKB stability theory.
- *Nonlinear effects of linear closure.* In principle, the Lagrangian statistics for the coefficients in the nonlinear $g\ell m$ equations may be closed at second moments, since the fluctuations ξ and $\mathbf{u}'$ are both taken to satisfy *linear* equations. Thus, the linearity of the $g\ell m$ equations would allow one to derive a set of equations for second moments such as $\overline{\xi \times \mathrm{curl}\,\mathbf{u}'}$ in the incompressible case and treat the combined system for the motion and the Lagrangian statistics as an initial value problem. This could be done by computing $\partial_t(\overline{\xi \times \mathrm{curl}\,\mathbf{u}'})$ using the linearized Euler motion equation for evolving $\mathbf{u}'$ and using the $\mathbf{u}'$ equation for the evolution of ξ.

  Such a linear closure would not produce only linear effects in the mean motion equation. The $g\ell m$ effects arise from second moments. The $g\ell m$ effects appear multiplicatively in the stress tensor and additively in the definition of the total mean momentum. The latter appears also in the nonlinearity of the $g\ell m$ equations. Thus, although the mean advection relations are enforced only at *linear* order, the contributions of the fluctuations to the $g\ell m$ motion equation are both linear and nonlinear.
- The $g\ell m$ theory expresses wave properties in terms of Lagrangian displacement statistics and gradients of mean flow properties. Consequently, one may consider imposing aspects of these relations between wave properties and

mean gradients, before taking variations in Hamilton's principle, by regarding these wave, mean flow relations as a type of *Taylor hypothesis*. We shall follow this idea further in Section 5.

4.6. EP $g\ell m$ equations for incompressible mean flow

The variational derivatives of the $g\ell m$ Lagrangian (4.5) for incompressible flow

$$\bar{\ell}(\bar{\mathbf{u}}, \bar{D}) = \int \left[\frac{1}{2}\bar{D}(|\bar{\mathbf{u}}|^2 + \overline{|\mathbf{u}'|^2}) + \bar{p}(1-\bar{D})\right] \mathrm{d}^3x, \tag{4.10}$$

are given by, cf. Eq. (4.4),

$$\delta\bar{\ell}(\bar{\mathbf{u}}, \bar{D}) = \int \left[\delta\bar{D}\left(\frac{1}{2}(|\bar{\mathbf{u}}|^2 + \overline{|\mathbf{u}'|^2}) - \bar{p}\right) + \delta\bar{p}(1-\bar{D}) + \bar{D}\delta\bar{\mathbf{u}}\cdot(\bar{\mathbf{u}} - \overline{\xi \times \mathrm{curl}\,\mathbf{u}'} + \nabla\overline{(\xi\cdot\mathbf{u}')}) + \delta\bar{\mathbf{u}}\cdot\overline{\mathbf{u}'\mathrm{div}(\bar{D}\xi)}\right] \mathrm{d}^3x.$$

We define the $g\ell m$ circulation velocity as

$$\bar{\mathbf{v}} \equiv \bar{\mathbf{u}} - \overline{\xi \times \mathrm{curl}\,\mathbf{u}'} + \nabla\overline{(\xi\cdot\mathbf{u}')}.$$

The corresponding EP motion equation (with $\nabla\cdot\bar{\mathbf{u}} = 0$) is expressed as

$$\frac{\partial}{\partial t}\bar{\mathbf{v}} + \bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{v}} + \bar{v}_j\nabla\bar{u}^j + \nabla\bar{p} = 0.$$

This is the EP equation for the Lagrangian (4.10). It also has the equivalent form

$$\frac{\partial}{\partial t}\bar{\mathbf{v}} - \bar{\mathbf{u}}\times\mathrm{curl}\,\bar{\mathbf{v}} + \nabla(\bar{\mathbf{v}}\cdot\bar{\mathbf{u}} + \bar{p}) = 0.$$

Thus, the Kelvin circulation theorem for the incompressible $g\ell m$ equations is simply

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c(\bar{\mathbf{u}})}(\bar{\mathbf{u}} - \overline{\xi\times\mathrm{curl}\,\mathbf{u}'})\cdot\mathrm{d}\mathbf{x} = 0. \tag{4.11}$$

Remark. We recall that $\mathbf{u}' = \partial_t\xi + \bar{\mathbf{u}}\cdot\nabla\xi - \xi\cdot\nabla\bar{\mathbf{u}}$. For the case that $\nabla\cdot\bar{\mathbf{u}} = 0$ and $\nabla\cdot\xi = 0$, this becomes $\mathbf{u}' = \partial_t\xi - \mathrm{curl}(\bar{\mathbf{u}}\times\xi)$. Hence, to *close* the $g\ell m$ EP motion equation for incompressible Eulerian mean flow, only *one key element* from the Lagrangian statistics must be specified. Namely, the quantity

$$\overline{\xi\times\omega'} = \overline{\xi\times\mathrm{curl}\,\mathbf{u}'} = \overline{\xi\times\mathrm{curl}\,(\partial_t\xi - \mathrm{curl}(\bar{\mathbf{u}}\times\xi))}, \tag{4.12}$$

must be specified in terms of $\bar{\mathbf{u}}$, $\nabla\bar{\mathbf{u}}$ and $\nabla\nabla\bar{\mathbf{u}}$. This specification is one of the main objectives of the discussions in the next section.

5. Alpha models

5.1. Opening remarks

We have seen that the use of Taylor expansions in the linearized fluctuation relations summons gradients of Eulerian mean fluid quantities into the mean second-variation Lagrangian. In turn, these gradients summon second-order spatial derivatives such as $\nabla\nabla\bar{\mathbf{u}}$ into the $g\ell m$ motion equation that results from the EP variational principle.

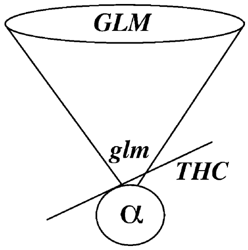


Fig. 1. GLM/$g\ell m$/THC technique.

Among other things, the $g\ell m$ stress tensor (4.9) for a compressible fluid shows the variety of channels available for energy exchange to occur. These channels arise through the various combinations of Eulerian mean gradients that appear in the stress tensor for the $g\ell m$ theory. The incompressible case is more straightforward because it has fewer such channels. However, to achieve closure, even the incompressible $g\ell m$ case still requires an assumption to express the key element of the Lagrangian statistics (4.12) in terms of $\bar{\mathbf{u}}$, $\nabla\bar{\mathbf{u}}$ and $\nabla\nabla\bar{\mathbf{u}}$. The linearized fluctuation equations themselves (relating the Eulerian and Lagrangian small disturbances) shall guide the formulation of such approximate closure assumptions.

Approach. The approach to the alpha-model equations is closely related to the $g\ell m$ approach, but with one important difference. Namely, the order is interchanged in the steps of making approximations and varying the EP Lagrangian in Hamilton's principle.

- *To obtain the $g\ell m$ equations:* we (i) expanded the Lagrangian, (ii) took its Eulerian mean, then (iii) varied to obtain the equations of motion, and *finally* saw the need to approximate the closure. This could be done, in principle, by using the $\mathbf{u}'$ equation for the tendency of ξ and the linearization of the GLM equations for the tendency of $\mathbf{u}'$. We shall discuss a more direct approach to closure that yields the α-models.
- *To obtain the α-models:* we shall (i) expand the Lagrangian, (ii) take its Eulerian mean, (iii) approximate the Lagrangian (by taking a particular solution of the $\mathbf{u}'$ equation as a *Taylor hypothesis*), and then (iv) vary to find a closed set of EP motion equations. This approach is illustrated in Fig. 1.

Remark. Because of the close relation between the approaches used in deriving these two sets of equations, one might hope for a bridge between them. For example, the $g\ell m$ equations could potentially provide an Eulerian diagnostic for determining parameters in the alpha model from DNS of the full Euler equations (or Navier–Stokes equations). The $g\ell m$ equations form a systematic approximation for the original GLM equations, within the EP framework. Thus, perhaps the GLM equations could be used to help answer questions that may arise at the other levels of approximation in this framework, particularly, in the alpha models.

5.2. THC approach

We shall use partial, or particular, solutions of the linearized velocity fluctuation equation (3.1)

$$\mathbf{u}' = \partial_t \xi + \bar{\mathbf{u}} \cdot \nabla \xi - \xi \cdot \nabla \bar{\mathbf{u}}$$

to guide certain choices of *Taylor hypotheses*. The three Taylor hypotheses we shall discuss for the $\mathbf{u}'$ equation are:[2]

[2] These three THC are for the $g\ell m$ equations. Later, we shall mention THC#4—for the GLM equations.

THC#1. Neglect space and time derivatives of ξ in the $\mathbf{u}'$ equation, or, set $\mathrm{d}\xi/\mathrm{d}t = \partial_t\xi + \bar{\mathbf{u}}\cdot\nabla\xi = 0$. In the incompressible case, this yields the original Euler-alpha model of Holm et al. [25,26], in which one assumes $\mathbf{u}' = -\xi\cdot\nabla\bar{\mathbf{u}}$.

THC#2. Assume that ξ is frozen-in as a one-form. In the incompressible case, this yields the anisotropic alpha model of Marsden and Shkoller [32], with $\mathbf{u}' = -2\xi\cdot\bar{\mathbf{e}}$, where $\bar{\mathbf{e}} = \frac{1}{2}(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^{\mathrm{T}})$ is the mean strain rate tensor, and $\partial_t\xi + \bar{\mathbf{u}}\cdot\nabla\xi = -\nabla\bar{\mathbf{u}}^{\mathrm{T}}\cdot\xi$.

THC#3. Assume that ξ is frozen-in as a two-form in three dimensions. Hence, set $\partial_t\xi + \bar{\mathbf{u}}\cdot\nabla\xi = \xi\cdot\nabla\bar{\mathbf{u}} - \xi\,\mathrm{div}\,\bar{\mathbf{u}}$. This assumption implies $\mathbf{u}' = -\xi\,\mathrm{div}\,\bar{\mathbf{u}}$, which, of course, is only interesting in the compressible case. For compressible flows, this choice lead to similarities with the Green–Naghdi equation for shallow water dynamics. The Green–Naghdi equation is discussed in [19]. See, e.g. [9] for more references and an asymptotic treatment of these equations.

Remark about algebraic closure. Neglecting the partial time derivative in the linearized velocity fluctuation equation for $\mathbf{u}'$ leads to an "algebraic closure relation," expressed as

$$\mathbf{u}' = \bar{\mathbf{u}}\cdot\nabla\xi - \xi\cdot\nabla\bar{\mathbf{u}} = \mathrm{curl}(\xi\times\bar{\mathbf{u}})$$

in the incompressible case. We note that substituting this algebraic closure relation into the $g\ell m$ Lagrangian (4.2) yields the following contribution to the mean fluctuational momentum

$$\overline{\mathbf{m}''} = \frac{\delta}{\delta\bar{\mathbf{u}}}\int\frac{1}{2}\overline{|\mathbf{u}'|^2}\,\mathrm{d}^3x = -\overline{\xi\times\mathrm{curl}\,\mathrm{curl}(\xi\times\bar{\mathbf{u}})}.$$

So, in this incompressible case, the contribution of the fluctuations to the mean total momentum (or pseudomomentum) keeps the same $g\ell m$ form as in Eqs. (4.11) and (4.12). However, the Lagrangian statistics is *steady* in Eulerian space. We shall decline this option.

The Taylor hypotheses THC#1–#3are approximate relations between Eulerian and Lagrangian statistics (namely, they are relations for $\mathbf{u}'$ as a function of ξ and its derivatives) that yield closures when substituted into the averaged Lagrangian in Hamilton's principle. We shall first discuss the incompressible case, which is simpler, and then we shall discuss the barotropic compressible case.

In both cases, we shall illustrate the *THC technique* by substituting the first of these three Taylor hypotheses into the $g\ell m$ Lagrangian, before taking its variations. This approach results via the EP framework in closed equations based on the $g\ell m$ equations that retain their Kelvin circulation theorem and conservation properties. Among these closed equations for the incompressible case are variants of the Euler-alpha model (or, averaged Euler equations) that are also related to the theory of second grade fluids and have been discussed as potential turbulence closure models when Navier–Stokes viscous dissipation is introduced, as in [10–13]. We shall show how this approach also leads to a new generalization of the Euler-alpha model that includes compressibility.

5.3. A brief history of the alpha models

The Euler-alpha equations for averaged incompressible ideal fluid motion were first derived in [25] in the context of the EP theory for fluid dynamics. That derivation proceeded essentially by choosing the kinetic energy to be the H_1 norm of the Eulerian fluid velocity, rather than the usual L_2 norm. This choice generalized the unidirectional shallow water equation of Camassa and Holm [8] from one dimension to three dimensions. The resulting n-dimensional Euler-alpha equation is (with $\nabla\cdot\mathbf{u} = 0$, $\mathbf{v}\equiv\mathbf{u} - \alpha^2\Delta\mathbf{u}$ and constant length scale α)

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v} + v_j\nabla u^j + \nabla p = 0. \qquad (5.1)$$

This is the EP equation for the Lagrangian[3]

$$\ell = \frac{1}{2}\int |\mathbf{u}|^2 + \alpha^2|\nabla\mathbf{u}|^2\,\mathrm{d}^3x \tag{5.2}$$

for a constant α and divergenceless fluid velocity $\mathbf{u}$. Mathematically, this equation describes geodesic motion on the volume-preserving diffeomorphism group of $\mathbf{R}^3$ relative to the H_1 norm in a sense similar to the work of Arnold [4] and Camassa and Holm [14] in which the Euler equations are shown to describe geodesic motion on the same diffeomorphism group relative to the L_2 kinetic energy norm.

Remarkably, the H_1-geodesic Euler-alpha equation was later recognized as being identical to the well-known invisicid second grade fluid equations introduced by Rivlin and Ericksen [34], although of course these equations were derived from a completely different viewpoint. The differences in their derivations imply corresponding differences in the interpretations of the solutions of these equations in each of their contexts. In particular, the constant parameter alpha (a length scale) is interpreted differently in the two theories. In the Euler-alpha model, the parameter alpha is associated with the flow regime and, in numerical simulations, *alpha separates active and passive degrees of freedom*, as shown in [13]. (Physically, alpha is the smallest active length scale participating in the nonlinear interactions—so scales smaller than alpha are swept along by the larger ones.) This is in contrast to the theory of second grade fluids, where alpha is an equilibrium thermodynamic material parameter whose values are restricted by the Gibbs–Duhem inequality.

THC#4, a nonlinear GLM Taylor hypothesis. In [21], a nonlinear GLM Taylor hypothesis was introduced and applied for both compressible and incompressible flows. This Taylor hypothesis THC#4for GLM assumes that the Lagrangian displacement fluctuation ξ is frozen as a *Lagrangian vector field* into the *nonlinear* GLM flow. Namely (note $\bar{\mathbf{u}}^{\mathrm{L}}$ rather than $\bar{\mathbf{u}}$)

$$\mathbf{u}^{\ell} \equiv \partial_t\xi + \bar{\mathbf{u}}^{\mathrm{L}}\cdot\nabla\xi = \xi\cdot\nabla\bar{\mathbf{u}}^{\mathrm{L}}. \tag{5.3}$$

This THC#4is substituted directly into the GLM averaged Lagrangian, e.g. (2.2) or (2.7), *without* linearizing the fluctuation relations. One may then vary the Lagrangian in the EP framework, *without* making the small-amplitude approximation. This THC#4treats the fluctuating Lagrangian displacement ξ as a material property associated with the frozen-in GLM motion of a “cloud” of fluid parcels initially displaced from one another by the initial value of ξ, which is *not* taken to vanish in this case. Under the GLM dynamics, the assumed nonlinear frozen-in relation for THC#4implies additional flow stresses as each fluid parcel convects this material property.

The THC#4may appear formally similar to the others, especially to THC#1. However, THC#4differs fundamentally from the others by being imposed as a finite, rather than a small-amplitude, approximation. Thus, THC#4couples to the gradients of the full Lagrangian-mean velocity, rather than to the gradients of its Eulerian mean small-amplitude approximation. Of course, the other THC#1–#3could also be made at the nonlinear GLM level, without first making the linearized fluctuation hypotheses that lead to the $g\ell m$ theory. It turns out that THC#1leads to a trivial result in this case, and the other nonlinear Taylor hypotheses have not yet been analyzed at the GLM level. The implications and physical interpretations of the GLM results of THC#4are discussed in [21]. This includes discussions of an interesting duality between the Eulerian-mean and Lagrangian-mean fluid velocities that arises for THC#4in the GLM theory.

5.3.1. *Extensions of Euler-α*

The works of Marsden and Shkoller [31] used the EP framework to introduce a certain type of filtering—called “fuzzying”—into the Lagrangian. Applying the EP reduction theorem to the Lagrangian for “fuzzy flow” yielded

[3] For incompressible flow $\nabla\cdot\mathbf{u} = 0$ and constant α, one may replace $|\nabla\mathbf{u}|^2$ in this Lagrangian equivalently with $2\mathrm{tr}(\mathsf{e}\cdot\mathsf{e})$, where $\mathsf{e} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})$ is the strain rate.

an alternative formal derivation of the incompressible Euler-α model, as well as an anisotropic variant of it and the extension of that variant to Riemannian manifolds. These works also showed short time existence for solutions of the Euler-α model and these extensions, by establishing for it the analog of the Ebin–Marsden theorem for the incompressible Euler equations, proven in [14]. See also [35,36,30] for the corresponding existence results for the original Euler-alpha model of Holm et al. [25].

While these references do make use of the EP reduction theorem, they do not show that the reduced equations obtained from it would also result from applying the "fuzzy flow" averaging method directly to Euler's equations. The EP Averaging Result *guarantees* this result, however, when the GLM averaging method is applied. For example, to the extent that the fuzzy averaging used in [31] fails to possess the projection property, the resulting EP equations will fail to coincide with fuzzy average of the original equations. Investigations of the relation between Marsden–Shkoller fuzzy averaging and GLM averaging are underway.

5.4. Barotropic $\overline{g\ell m}$, a closure model for barotropic $g\ell m$

In seeking its variational closure, we shall start with the small-amplitude $g\ell m$ Lagrangian (4.2) for barotropic compressible flow at second order

$$\bar{\ell} = \bar{\ell}_0 + \frac{1}{2}\overline{\ell''} = \int\left[\frac{1}{2}\bar{D}|\bar{\mathbf{u}}|^2 - \bar{D}e(\bar{D})\right]\mathrm{d}^3x + \int\left[\frac{1}{2}\bar{D}\overline{|\mathbf{u}'|^2} + \overline{D'\mathbf{u}'}\cdot\bar{\mathbf{u}} - \frac{1}{2}\frac{c^2(\bar{D})}{\bar{D}}\overline{D'^2}\right]\mathrm{d}^3x. \tag{5.4}$$

Into this $g\ell m$ Lagrangian, we shall substitute the simplest available hypothesis for closing the barotropic $g\ell m$ system, namely[4]

$$\mathbf{u}' = -\xi\cdot\nabla\bar{\mathbf{u}} \quad \text{and} \quad D' = -\xi\cdot\nabla\bar{D}. \tag{5.5}$$

This substitution yields the mean Lagrangian for the closed barotropic $g\ell m$ system (barotropic $\overline{g\ell m}$)

$$\begin{aligned}\bar{\ell} = \bar{\ell}_0 + \frac{1}{2}\overline{\ell''} &= \int\left[\frac{1}{2}\bar{D}|\bar{\mathbf{u}}|^2 - \bar{D}e(\bar{D})\right]\mathrm{d}^3x \\ &\quad + \int\left[\frac{1}{2}\bar{D}\overline{\xi^k\xi^l}\bar{\mathbf{u}}_{,k}\cdot\bar{\mathbf{u}}_{,l} + \overline{\xi^k\xi^l}\bar{\mathbf{u}}\cdot\bar{\mathbf{u}}_{,k}\bar{D}_{,l} - \frac{c^2(\bar{D})}{2\bar{D}}\overline{\xi^k\xi^l}\bar{D}_{,k}\bar{D}_{,l}\right]\mathrm{d}^3x.\end{aligned} \tag{5.6}$$

With the $\overline{g\ell m}$ closure hypothesis (5.5) no derivatives of the fluctuation statistics appear in this mean Lagrangian.

Combining the closure hypothesis (5.5) with the $\mathbf{u}'$ equation (3.1) implies $(\partial_t + \bar{\mathbf{u}}\cdot\nabla)\xi = 0$, i.e., componentwise advection of ξ. Consequently, the components of the quadratic Lagrangian moments are simply carried along with the Eulerian mean flow, as

$$(\partial_t + \bar{\mathbf{u}}\cdot\nabla)\overline{\xi^k\xi^l} = 0. \tag{5.7}$$

This $\overline{\xi\xi}$ equation admits the *isotropic solution*

$$\overline{\xi^k\xi^l} = \alpha^2\delta^{kl},$$

where α is an advected scalar $(\partial_t + \bar{\mathbf{u}}\cdot\nabla)\alpha = 0$, that has dimensions of length. In turn, this advective relation for α also admits a constant solution, should we wish to simplify the dynamics of the Lagrangian moments even further.

[4] For compressible flows, the other Taylor hypotheses THC#2and THC#3lead to similar formulas to those given in this section. In particular, all three Taylor hypotheses preserve properties of homogeneity and isotropy, if these properties are initially present. We shall not discuss those other cases here. The implications of Taylor hypothesis THC#2for *incompressible* flows are analyzed in [32].

To develop the barotropic closure model $\overline{g\ell m}$, we shall substitute the variational derivatives of $\bar{\ell}$ into the following EP equation, cf. Eq. (2.1),

$$\frac{\partial}{\partial t}\frac{\delta\bar{\ell}}{\delta\bar{u}^i} + \frac{\partial}{\partial x^k}\left(\frac{\delta\bar{\ell}}{\delta\bar{u}^i}\bar{u}^k\right) + \frac{\delta\bar{\ell}}{\delta\bar{u}^k}\frac{\partial\bar{u}^k}{\partial x^i} = \bar{D}\frac{\partial}{\partial x^i}\frac{\delta\bar{\ell}}{\delta\bar{D}} - \frac{\delta\bar{\ell}}{\delta\overline{\xi^k\xi^l}}\frac{\partial\overline{\xi^k\xi^l}}{\partial x^i}. \tag{5.8}$$

The contribution of the last term arises from the scalar advection of the components of $\overline{\xi^k\xi^l}$. The necessary variational derivatives may be obtained from

$$\delta\bar{\ell} = \int\left[((1-\hat{\Delta})(\bar{D}\bar{\mathbf{u}}) + \overline{\xi^k\xi^l}\bar{\mathbf{u}}_{,k}\bar{D}_{,l})\cdot\delta\bar{\mathbf{u}} + \Gamma_{kl}\delta(\overline{\xi^k\xi^l}) - \bar{\Pi}^{g\ell m}\delta\bar{D}\right]\mathrm{d}^3x, \tag{5.9}$$

with homogeneous boundary conditions,

$$\hat{\mathbf{n}}\cdot\bar{\mathbf{u}} = 0 \quad\text{and}\quad \hat{\mathbf{n}}\cdot\overline{\xi\xi} = 0 \quad \text{on the boundary.}$$

These are the physically meaningful conditions at fixed boundaries. Weaker boundary conditions may also suffice in this case, namely,

$$\hat{\mathbf{n}}\cdot\bar{\mathbf{u}} = 0 \quad\text{and}\quad \hat{\mathbf{n}}\times(\hat{\mathbf{n}}\cdot\overline{\xi\xi}\cdot\nabla)\bar{\mathbf{u}} = 0 \quad \text{on the boundary.}$$

Here the generalized Laplacian operator $\hat{\Delta}$ is defined by

$$\hat{\Delta} = \partial_l\overline{\xi^k\xi^l}\partial_k, \tag{5.10}$$

and the $\overline{g\ell m}$ potential $\bar{\Pi}^{g\ell m}$ is defined by

$$\bar{\Pi}^{g\ell m} = (1-\hat{\Delta})(\tfrac{1}{2}|\bar{\mathbf{u}}|^2 - h(\bar{D})) + \tfrac{1}{2}\overline{\xi^k\xi^l}\bar{\mathbf{u}}_{,k}\cdot\bar{\mathbf{u}}_{,l} - \tfrac{1}{2}\overline{\xi^k\xi^l}\bar{D}_{,k}\bar{D}_{,l}h''(\bar{D}),$$

where $h'(\bar{D}) = c^2(\bar{D})/\bar{D}$. The quantity Γ_{kl} denotes the variational derivative of $\bar{\ell}$ with respect to the mean Lagrangian statistical moments. Namely,

$$\Gamma_{kl} = \frac{\delta\bar{\ell}}{\delta\overline{\xi^k\xi^l}} = \frac{1}{2}\bar{D}\bar{\mathbf{u}}_{,k}\cdot\bar{\mathbf{u}}_{,l} + \bar{\mathbf{u}}\cdot\bar{\mathbf{u}}_{,k}\bar{D}_{,l} - \frac{1}{2}\bar{D}_{,k}\bar{D}_{,l}h'(\bar{D}).$$

The EP motion equation (5.8) for the barotropic $\overline{g\ell m}$ closure model is, thus,

$$\partial_t\bar{m}_i + \partial_j(\bar{m}_i\bar{u}^j) + \bar{m}_j\partial_i\bar{u}^j = \bar{D}\partial_i\bar{\Pi}^{g\ell m} - \Gamma_{kl}\partial_i\overline{\xi^k\xi^l}, \tag{5.11}$$

where the total mean momentum for barotropic $\overline{g\ell m}$ is given by

$$\bar{\mathbf{m}} = \frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}} = (1-\hat{\Delta})(\bar{D}\bar{\mathbf{u}}) + \overline{\xi^k\xi^l}\bar{\mathbf{u}}_{,k}\bar{D}_{,l} = \bar{D}\bar{\mathbf{u}} - \frac{1}{2}[\hat{\Delta}(\bar{D}\bar{\mathbf{u}}) + \bar{D}\hat{\Delta}\bar{\mathbf{u}} + \bar{\mathbf{u}}\hat{\Delta}\bar{D}]. \tag{5.12}$$

This momentum may be expressed as an L^2-symmetric operator acting on the mean fluid velocity:

$$\bar{\mathbf{m}} = [(\bar{D} - \tfrac{1}{2}\hat{\Delta}\bar{D}) - \tfrac{1}{2}(\hat{\Delta}\bar{D}\,\cdot + \bar{D}\hat{\Delta}\,\cdot)]\bar{\mathbf{u}} \equiv (\bar{D} - \hat{\mathcal{O}})\bar{\mathbf{u}}, \tag{5.13}$$

which defines the operator $\hat{\mathcal{O}}$. (The first parenthesis in the square brackets contains a multiplier and the second one contains a L^2-symmetric operator.)

To make a connection between the barotropic $\overline{g\ell m}$ motion equation (5.11) and the original GLM motion equation (1.5), we shall identify the mean momentum as $\bar{\mathbf{m}} = \bar{D}(\bar{\mathbf{u}} - \bar{\mathbf{p}})$ with pseudomomentum density

$$\bar{D}\bar{\mathbf{p}} \equiv \tfrac{1}{2}[\hat{\Delta}(\bar{D}\bar{\mathbf{u}}) + \bar{D}\hat{\Delta}\bar{\mathbf{u}} + \bar{\mathbf{u}}\hat{\Delta}\bar{D}] = \hat{\mathcal{O}}\bar{\mathbf{u}}.$$

This identification of pseudomomentum and the continuity equation for $\bar{D}$ allows Eq. (5.11) to be rewritten as, cf. Eq. (1.5),

$$(\partial_t + \bar{\mathbf{u}}\cdot\nabla)(\bar{\mathbf{u}} - \bar{\mathbf{p}}) + (\bar{u}_k - \bar{p}_k)\nabla\bar{u}^k - \nabla\bar{\Pi}^{g\ell m} + \frac{1}{\bar{D}}\Gamma_{kl}\nabla\overline{\xi^k\xi^l} = 0. \tag{5.14}$$

The closed barotropic $\overline{g\ell m}$ system consists of the EP motion equation (5.11) and two auxiliary equations. These are the continuity equation (3.5) for $\bar{D}$ and the advection equation (5.7) for $\overline{\xi^k\xi^l}$, recalled as

$$\partial_t\bar{D} + \operatorname{div}\bar{D}\bar{\mathbf{u}} = 0 \quad \text{and} \quad (\partial_t + \bar{\mathbf{u}}\cdot\nabla)\overline{\xi^k\xi^l} = 0. \tag{5.15}$$

The dynamical properties of the closed barotropic $\overline{g\ell m}$ system may be investigated using the EP framework. For these equations, we have the Kelvin–Noether circulation theorem, as well as conservation laws for potential vorticity, momentum and energy.

5.4.1. *Kelvin–Noether circulation theorem for barotropic $\overline{g\ell m}$*

In the EP framework, the Kelvin–Noether theorem implies the circulation relation, cf. Eq. (2.2) for adiabatic GLM,

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c(\bar{\mathbf{u}})}\frac{1}{\bar{D}}\bar{\mathbf{m}}\cdot\mathrm{d}\mathbf{x} = -\oint_{c(\bar{\mathbf{u}})}\frac{1}{\bar{D}}\Gamma_{kl}\,\mathrm{d}\overline{\xi^k\xi^l}.$$

Thus, the advected Lagrangian statistical moments $\overline{\xi^k\xi^l}$ play the same role that specific entropy and relative buoyancy played in the adiabatic and stratified GLM cases treated earlier. From Stokes theorem and scalar advection of $\overline{\xi^k\xi^l}$, we also find the *tensor potential vorticity evolution* equation

$$(\partial_t + \bar{\mathbf{u}}\cdot\nabla)q^{pq} = -\nabla\left(\frac{1}{\bar{D}}\Gamma_{kl}\right)\cdot\nabla\overline{\xi^k\xi^l}\times\nabla\overline{\xi^p\xi^q}, \quad \text{where } q^{pq} = \frac{1}{\bar{D}}\nabla(\overline{\xi^p\xi^q})\cdot\operatorname{curl}\left(\frac{1}{\bar{D}}\bar{\mathbf{m}}\right), \quad \forall p,q.$$

Thus, anisotropy in the statistics of $\overline{\xi^k\xi^l}$ will create tensor potential vorticity q^{pq}. If the fluctuation statistics are initially isotropic, so that $\overline{\xi^k\xi^l} = \alpha^2\delta^{kl}$ at the initial time, then they will remain so, and the correspondingly isotropic potential vorticity will be conserved along barotropic $\overline{g\ell m}$ particle trajectories.

5.4.2. *Momentum conservation for barotropic $\overline{g\ell m}$*

Because the Lagrangian $\bar{\ell}$ in Eq. (5.6) is invariant under translations, Noether's theorem yields the momentum conservation law (2.5),

$$\partial_t\bar{m}_i + \partial_j\bar{T}_i^j = 0,$$

where $\bar{\mathbf{m}} = \delta\bar{\ell}/\delta\bar{\mathbf{u}}$ is the $\overline{g\ell m}$ Eulerian-mean momentum density in Eq. (2.6) and the Eulerian-mean stress tensor $\bar{T}_i^j$ is written in Eq. (2.6) in the form

$$\bar{T}_i^j = \bar{m}_i\bar{u}^j + \delta_i^j\left(\bar{\mathcal{L}} - \bar{D}\frac{\partial\bar{\mathcal{L}}}{\partial\bar{D}}\right).$$

For the $\overline{g\ell m}$ theory, this *stress tensor* is given in terms of mean fluid quantities by

$$\bar{T}_i^j = \bar{m}_i\bar{u}^j + \delta_i^j\mathcal{P} - \bar{D}_{,i}\overline{\xi^j\xi^k}(|\bar{\mathbf{u}}|^2 - h(\bar{D}))_{,k} - \bar{u}^m_{,i}\overline{\xi^j\xi^k}(\bar{D}\bar{u}_m)_{,k}.$$

Here $\mathcal{P}$ denotes the total $\overline{g\ell m}$ *mean pressure*,

$$\mathcal{P} = (1-\hat{\Delta})p(\bar{D}) + \tfrac{1}{2}\bar{D}\hat{\Delta}|\bar{\mathbf{u}}|^2 + \tfrac{1}{2}\overline{\xi^k\xi^l}\bar{D}_{,l}(|\bar{\mathbf{u}}|^2 + c^2(\bar{D}) + h(\bar{D}))_{,k}.$$

For an ideal γ-law gas, $c^2 = \gamma p(\bar{D})/\bar{D}$ and $c^2 + h = \gamma c^2/(\gamma - 1)$.

5.4.3. Energy conservation for barotropic $\overline{g\ell m}$

The Legendre transformation of the mean Lagrangian (5.6) yields the conserved mean energy, also given by Noether's theorem, as

$$\bar{\mathcal{H}} = \int \frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}}\cdot\bar{\mathbf{u}}\,\mathrm{d}^3x - \bar{\ell}(\bar{\mathbf{u}}, \bar{D}, \overline{\xi^k\xi^l}) = \int\left[\frac{1}{2}\bar{D}|\bar{\mathbf{u}}|^2 + \bar{D}e(\bar{D}) + \frac{1}{2}\bar{D}\overline{\xi^k\xi^l}\bar{\mathbf{u}}_{,k}\cdot\bar{\mathbf{u}}_{,l} + \frac{c^2(\bar{D})}{2\bar{D}}\overline{\xi^k\xi^l}\bar{D}_{,k}\bar{D}_{,l}\right]\mathrm{d}^3x.$$

We note that we may write the latter two terms in the mean conserved energy $\bar{\mathcal{H}}$, i.e., those due only to fluctuations, as

$$\frac{1}{2}\overline{\mathcal{H}''} = \int\left[\frac{1}{2}\bar{D}\overline{|\mathbf{u}'|^2} + \frac{c^2(\bar{D})}{2\bar{D}}\overline{D'^2}\right]\mathrm{d}^3x \neq \frac{1}{2}\overline{E''}.$$

This expression does *not* recover the result of Blokhintsev [5] mentioned earlier in Eq. (4.4). However, it has the advantage of being a positive-definite mean fluctuational energy for the closed $\overline{g\ell m}$ system.

5.4.4. Lie–Poisson Hamiltonian formulation of barotropic $\overline{g\ell m}$

Being an EP system, the barotropic $\overline{g\ell m}$ theory may be transformed into Lie–Poisson Hamiltonian form, by following the procedure explained in [25]. This Hamiltonian formulation begins by writing the Legendre-transformed energy in terms of the momentum. We shall assume the operator $(\bar{D}-\hat{\mathcal{O}})$ in momentum–velocity relation (5.13) is invertible, so that one may solve for the velocity from the momentum as $\bar{\mathbf{u}} = (\bar{D}-\hat{\mathcal{O}})^{-1}\bar{\mathbf{m}}$. Thus, the energy Hamiltonian for

$$\bar{\mathcal{H}} = \int\left[\frac{1}{2}\bar{\mathbf{m}}\cdot(\bar{D}-\hat{\mathcal{O}})^{-1}\bar{\mathbf{m}} + \bar{D}e(\bar{D}) + \frac{c^2(\bar{D})}{2\bar{D}}\overline{\xi^k\xi^l}\bar{D}_{,k}\bar{D}_{,l}\right]\mathrm{d}^3x.$$

The ideal barotropic $\overline{g\ell m}$ equations may now be treated in the Lie–Poisson Hamiltonian framework, if so desired. The corresponding Lie–Poisson bracket is of the standard type, defined on the dual of a certain semidirect-product Lie algebra, as described, e.g., in [28]. See also [29] for an introduction to this now-standard theory and references to the literature.

5.4.5. Barotropic $\overline{g\ell m}$-α, a simplification of barotropic $\overline{g\ell m}$ for constant isotropic Lagrangian statistics

The scalar advection equation (5.15) for the Lagrangian statistical moments admits the constant isotropic solution $\overline{\xi^k\xi^l} = \alpha^2\delta^{kl}$, where α is a constant length scale. The $\overline{g\ell m}$ Lagrangian (5.6) in this case simplifies to, cf. the Lagrangian (5.2) for the incompressible Euler-alpha model:

$$\bar{\ell} = \int\left[\frac{1}{2}\bar{D}|\bar{\mathbf{u}}|^2 - \bar{D}e(\bar{D})\right]\mathrm{d}^3x + \alpha^2\int\left[\frac{1}{2}\bar{D}|\nabla\mathbf{u}|^2 + \frac{1}{2}\nabla|\bar{\mathbf{u}}|^2\cdot\nabla\bar{D} - \frac{c^2(\bar{D})}{2\bar{D}}|\nabla\bar{D}|^2\right]\mathrm{d}^3x. \qquad (5.16)$$

This is the Lagrangian for the compressible $\overline{g\ell m}$-α model with constant length scale α. For constant α, the generalized Laplacian $\hat{\Delta}$ in the previous equations reduces to $\hat{\Delta}\to\alpha^2\Delta$, where Δ is the ordinary Laplacian. The result is a *compressible generalization of the Euler-alpha model*. The equations of motion for this model are (5.11), (5.13) and (5.15) with $\overline{\xi^k\xi^l} = \alpha^2\delta^{kl}$ and $\hat{\Delta}\to\alpha^2\Delta$ for constant α.

5.4.6. Barotropic $g\ell m$ models in a rotating frame

The EP setting is convenient for transforming the averaged Lagrangian $\bar{\ell}$ into a rotating frame. One defines the rotation vector potential $\mathbf{R}$ satisfying $\mathrm{curl}\,\mathbf{R} = 2\Omega(\mathbf{x})$, for a spatially dependent rotation frequency $\Omega(\mathbf{x})$. The transformation begins by substituting into the original Lagrangian the linearized relation $\bar{\mathbf{u}} + \mathbf{u}' = \bar{\mathbf{u}}^* + \mathbf{u}'^* + \bar{\mathbf{R}} + \mathbf{R}'$, with $\mathbf{R}' = \mathbf{R}^\ell - \xi\cdot\nabla\bar{\mathbf{R}}$. One then averages and finally drops the asterisk in $(\cdot)^*$

to find, cf. Eq. (5.4),

$$\bar{\ell} = \int \left[\frac{1}{2} \bar{D} |\bar{\mathbf{u}} + \bar{\mathbf{R}}|^2 - \bar{D} e(\bar{D}) \right] \mathrm{d}^3 x + \int \left[\frac{1}{2} \bar{D} \overline{|\mathbf{u}'|^2} + \frac{1}{2} \bar{D} \overline{|\mathbf{R}'|^2} + \overline{D'(\mathbf{u}' + \mathbf{R}')} \cdot (\bar{\mathbf{u}} + \bar{\mathbf{R}}) - \frac{1}{2} \frac{c^2(\bar{D})}{\bar{D}} \overline{D'^2} \right] \mathrm{d}^3 x. \quad (5.17)$$

For a *constant* rotation frequency, $\bar{\mathbf{R}} = \Omega \times \mathbf{x}$ and $\mathbf{R}'$ vanishes. In this Lagrangian, the velocities $\bar{\mathbf{u}}$ and $\mathbf{u}'$ are measured in the rotating frame. The analysis then proceeds as in the earlier sections in the EP setting. See, e.g. Eqs. (2.2), (2.7), (4.2), (5.6) and (5.8).

Outlook.

- We shall discuss the effects of rotation and its interaction with fluctuations elsewhere, in the simpler shallow water context where we may also compare with results of Bühler and McIntyre [6].
- The proper choice of dissipation for this system deserves further investigation. One may propose to add dissipation as *shear and bulk viscosity* in the form

$$\partial_t, \bar{m}_i + \partial_j \bar{T}_i^j = \frac{\partial}{\partial x^j} (\mu \hat{\mathcal{O}}(u_{i,j} + u_{j,i}) + \eta \delta_{ij} \hat{\mathcal{O}}(u_{k,k})),$$

where $\hat{\mathcal{O}}$ is the positive L^2-symmetric operator in Eq. (5.13). This choice assures monotonic decay of the energy in (5.16) for homogeneous boundary conditions. Work is in progress to add viscosity and account for its heating effects in compressible flows. However, these effects go beyond the $\overline{g\ell m} - \alpha$ equations and will be discussed elsewhere. For discussions of viscous effects in the incompressible Lagrangian Averaged Navier–Stokes-α equations, see [15,16,31].

6. Conclusions

Synopsis. This paper connects the GLM equations to the Lagrangian-averaged Euler-alpha models through a new set of Eulerian mean $g\ell m$ fluid equations that are derived in the small-amplitude limit of the GLM equations. These equations comprise a one-point closure approximation that is second-order in the Lagrangian fluctuation statistics. In principle, these equations may be closed by using the linearized dynamics of the original equations in combination with the linearized fluctuation relations. However, because of the complexity still remaining even at the intermediate $g\ell m$ level, we sought simpler closures by using these linearized fluctuation relations to guide our choices among the various *Taylor hypotheses* for deriving variants of the Euler-alpha models and related models in a variational closure procedure. Following the original procedure discussed in [25] for deriving the Euler-alpha model, one substitutes these linear versions of the Taylor hypothesis into the Lagrangian before taking its Eulerian mean and then its variations in the EP framework. This procedure preserved the Kelvin–Noether circulation theorem, which is a basic geometrical property of all ideal fluid models. This procedure also preserved the mean momentum and energy balances. Finally, the procedure led to a barotropic compressible generalization of the Euler-alpha models.

6.1. Summary

In this paper, we developed the geometric approach to dimension reduction especially for models of turbulence in weakly compressible fluids in the context of GLM averaging. Our approach concentrated on reduction of the Lagrangian in Hamilton's principle for adiabatic compressible fluid dynamics by using a combination of compatible symmetries and averaging in the EP framework. This approach is versatile enough to include ocean circulation models for global climate modeling, as well as fundamental research in turbulence. The present paper analyzes the basic equations in the framework of the EP theory and thereby presents them in a unified geometrical context for further application.

The EP Averaging Result establishes the equivalence of modeling using the GLM approach, either by directly averaging the equations of motion, or by averaging the Lagrangian for these equations before taking its variations. We discussed EP formulations of both Lagrangian-mean, and Eulerian-mean fluid equations for modeling turbulence.

We used various elements of the classical theory of turbulence, including:

- Reynolds decomposition(s),
- THC (Taylor Hypothesis Closure),
- Hamilton's principle,
- Averaged Lagrangians and
- Euler–Poincaré equations

to model and analyze the mean dynamical effects of fluctuations on three-dimensional exact Lagrangian-mean and approximate second-order Eulerian-mean fluid motion.

Our starting point was the exact nonlinear GLM equations of Andrews and McIntyre [2] for a compressible adiabatic fluid. We first recast the GLM equations as EP equations resulting from the Lagrangian mean of Hamilton's principle, written in the Eulerian fluid description. This demonstrated the validity of the general principles underlying the EP Averaging Result. We then used the small-amplitude approximation to linearize the relations between Lagrangian disturbances and Eulerian fluctuations. We substituted these linearizations into Hamilton's principle for the GLM equations and kept terms up to quadratic order before taking the Eulerian mean. The EP equations resulting from this approximate Eulerian-mean Lagrangian produced a new set of $g\ell m$ equations. These comprise a second-order (one-point, weakly compressible) turbulence closure model that captures some aspects of the influence of the small scale dynamics on the large scale flow—while preserving the mathematical structure of the original Euler equations.

We observed that $g\ell m$ theory relates certain combined Eulerian and Lagrangian aspects of wave properties through expressions also involving gradients of mean flow properties. This observation suggested we consider closure schemes that involve substituting approximations or truncated versions of these relations between wave properties and mean gradients into Hamilton's principle, before taking its variations. Thus, we regarded these approximated, or truncated, relations as a type of *THC*. We tried one of the simpler variants of this idea and found a new compressible generalization of the Euler-alpha models, the $\overline{g\ell m}$ closure, whose solution behavior remains to be studied.

Thus, introducing such Taylor-hypotheses into the second-order Eulerian-mean closure approximation for the $g\ell m$ theory led to variants of the Euler-alpha models, and a framework for exploring other options. This included finding new variants of them for compressible flows that we discussed in Section 5.4.

Being derivable in the EP framework, the GLM theory, as well as its second-order Eulerian-mean closure approximation, the new $g\ell m$ theory, and the new compressible $\overline{g\ell m}$ generalization of the Euler-alpha equations, all possess the same fundamental structure and underlying geometry that are shared by all other ideal fluid theories in the EP framework. This geometrical structure ensures that these fluid theories (both exact and approximate ones) each retains its own Kelvin circulation theorem and the associated conservation law for potential vorticity arising from it by Noether's theorem, for particle relabeling symmetry. The EP framework also implies balance laws for momentum and energy exchanges between mean flow and wave properties.

The geometrical structure of the EP framework leads, in addition, to the Lie–Poisson Hamiltonian formulation for GLM theory, its Eulerian-mean closure approximation and the variants of the Euler-alpha models. This Hamiltonian formulation possesses potential-vorticity Casimirs associated with its Lie–Poisson bracket. In turn, the Lie–Poisson Hamiltonian structure leads to the energy-Casimir method for characterizing equilibrium solutions as critical points of a constrained energy and for establishing their nonlinear Liapunov stability conditions. All of these additional features are now available, for the GLM theory, for its Eulerian-mean closure approximation, the $g\ell m$ theory, for the compressible $\overline{g\ell m}$ closure model, and also for the alpha models and any new variants of them that may arise in the future.

The $g\ell m$ theory provides a bridge that spans from the alpha models to the exact nonlinear GLM theory. We hope this bridge will be useful in answering questions that arise in the context of the alpha models and other turbulence closure models. Of course, much remains to be done in this regard. The $g\ell m$ framework seems to offer a promising new opportunity for modeling the nonlinearity of fluid turbulence from a Lagrangian perspective.

Acknowledgements

We are grateful for stimulating discussions of this topic with P. Constantin, G. Eyink, J.E. Marsden, M.E. McIntyre and I. Mezic. Some of these discussions took place at Cambridge University while the author was a visiting professor at the Isaac Newton Institute for Mathematical Science. We are also grateful to R. Lowrie, L. Margolin, R. Ristorcelli, D.M. Tartakovsky, B. Wingate, L. Winter and others in the Los Alamos Turbulence Working Group for many worthwhile discussions. This work was supported by the US Department of Energy under contracts W-7405-ENG-36 and the Applied Mathematical Sciences Program KC-07-01-01.

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