A Quick Note on the Anelastic Spherical Harmonic (ASH) Code

Allan Sacha Brun

based on Brun, Miesch, Toomre 2004, ApJ, 614, 1073,
and Clune et al. 1999, Parallel Computing, 25,361

1. Formulation of the Anelastic MHD Equations in the ASH Code

In this paper we report three-dimensional numerical experiments designed to investigate the complex magnetohydrodynamics (MHD) of the solar convection zone in spherical geometries. We have extended our already well-tested hydrodynamic ASH code (anelastic spherical harmonic; see Clune et al. 1999, Miesch et al. 2000, Brun & Toomre 2002) to include the magnetic induction equation and the feedback of the field on the flow via Lorentz forces and ohmic heating. Thus, the ASH code is now able to solve the full set of 3–D MHD anelastic equations of motion in a rotating, convective spherical shell (Glatzmaier 1984) with high resolution on massively-parallel computing architectures. These equations are fully nonlinear in velocity and magnetic field variables, but under the anelastic approximation the thermodynamic variables are linearized with respect to a spherically symmetric and evolving mean state having a density $\bar{\rho}$, pressure $\bar{P}$, temperature $\bar{T}$ and specific entropy $\bar{S}$. Fluctuations about this mean state are denoted by ρ, P, T, and S. The resulting equations are:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\bar{\rho}\mathbf{v}) &= 0, && (1)\\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0, && (2)\\
\bar{\rho}\left(\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+2\boldsymbol{\Omega}_{\mathbf{o}}\times\mathbf{v}\right) &= -\boldsymbol{\nabla}P+\rho\mathbf{g}+\frac{1}{4\pi}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}\\
&- \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{D}}-[\boldsymbol{\nabla}\bar{P}-\bar{\rho}\mathbf{g}], && (3)\\
\bar{\rho}\bar{T}\frac{\partial S}{\partial t}+\bar{\rho}\bar{T}\mathbf{v}\cdot\boldsymbol{\nabla}(\bar{S}+S) &= \boldsymbol{\nabla}\cdot[\kappa_r\bar{\rho}c_p\boldsymbol{\nabla}(\bar{T}+T)+\kappa\bar{\rho}\bar{T}\boldsymbol{\nabla}(\bar{S}+S)]\\
&+ \frac{4\pi\eta}{c^2}\mathbf{j}^2+2\bar{\rho}\nu\left[e_{ij}e_{ij}-1/3(\boldsymbol{\nabla}\cdot\mathbf{v})^2\right]+\bar{\rho}\epsilon, && (4)\\
\frac{\partial\mathbf{B}}{\partial t} &= \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B})-\boldsymbol{\nabla}\times(\eta\boldsymbol{\nabla}\times\mathbf{B}), && (5)
\end{aligned}$$

where $\mathbf{v}=(v_r,v_\theta,v_\phi)$ is the local velocity in spherical coordinates in the frame rotating at constant angular velocity $\boldsymbol{\Omega}_{\mathbf{o}}$, $\mathbf{g}$ is the gravitational acceleration, $\mathbf{B}=(B_r,B_\theta,B_\phi)$ is the magnetic field, $\mathbf{j}=c/4\pi\,(\boldsymbol{\nabla}\times\mathbf{B})$ is the current density, c_p is the specific heat at constant pressure, κ_r is the radiative diffusivity, η is the effective magnetic diffusivity, and $\mathcal{D}$ is the viscous stress tensor, involving the components

$$\mathcal{D}_{ij} = -2\bar{\rho}\nu[e_{ij} - 1/3(\boldsymbol{\nabla} \cdot \mathbf{v})\delta_{ij}], \tag{6}$$

where e_{ij} is the strain rate tensor, and ν and κ are effective eddy diffusivities. A volume heating term $\bar{\rho}\epsilon$ is also included in these equations for completeness but it is insignificant in the solar envelope. When our model is applied to other stars, such as A-type stars (Browning, Brun & Toomre 2004), this term represents energy generation by nuclear burning. To complete the set of equations, we use the linearized equation of state

$$\frac{\rho}{\bar{\rho}} = \frac{P}{\bar{P}} - \frac{T}{\bar{T}} = \frac{P}{\gamma\bar{P}} - \frac{S}{c_p}, \tag{7}$$

where γ is the adiabatic exponent, and assume the ideal gas law

$$\bar{P} = \mathcal{R}\bar{\rho}\bar{T} \tag{8}$$

where $\mathcal{R}$ is the gas constant. The reference or mean state (indicated by overbars) is derived from a one-dimensional solar structure model (Brun et al. 2002) and is continuously updated with the spherically-symmetric components of the thermodynamic fluctuations as the simulation proceeds. It begins in hydrostatic balance so the bracketed term on the right-hand-side of equation (3) initially vanishes. However, as the simulation evolves, turbulent and magnetic pressure drive the reference state slightly away from hydrostatic balance.

Due to limitations in computing resources, no simulation achievable now or in the near future can hope to directly capture all scales of solar convection from global to molecular dissipation scales. The simulations reported here resolve nonlinear interactions among a larger range of scales than any previous MHD model of global-scale solar convection but motions still must exist in the sun on scales smaller than our grid resolution. In this sense, our models should be regarded as large-eddy simulations (LES) with parameterizations to account for subgrid-scale (SGS) motions. Thus the effective eddy diffusivities ν, κ, and η represent momentum, heat, and magnetic field transport by motions which are not resolved by the simulation. They are allowed to vary with radius but are independent of latitude, longitude, and time for a given simulation. Their amplitudes and radial profiles are varied depending on the resolution and objectives of each simulation. In the simulations reported here, ν, κ, and η are assumed to be proportional to $\bar{\rho}^{-1/2}$.

The velocity, magnetic, and thermodynamic variables are expanded in spherical harmonics $Y_{\ell m}(\theta, \phi)$ for their horizontal structure and in Chebyshev polynomials $T_n(r)$ for their radial structure (see Appendix A). This approach has the advantage that the spatial resolution is uniform everywhere on a sphere when a complete set of spherical harmonics is used up to some maximum in degree ℓ (retaining all azimuthal orders $m \leq \ell$ in what is known as triangular truncation).

The anelastic approximation captures the effects of density stratification without having to resolve sound waves which would severely limit the time step. In the MHD context, the anelastic approximation filters out fast magneto-acoustic waves but retains the Alfven and slow magneto-acoustic modes. In order to ensure that the mass flux and the magnetic field remain divergenceless to machine precision throughout the simulation, we use a toroidal–poloidal decomposition as:

$$\bar{\rho}\mathbf{v} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (W\hat{\mathbf{e}}_r) + \boldsymbol{\nabla} \times (Z\hat{\mathbf{e}}_r), \tag{9}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (C\hat{\mathbf{e}}_r) + \boldsymbol{\nabla} \times (A\hat{\mathbf{e}}_r) \quad . \tag{10}$$

Appendix A lists the full set of anelastic MHD equations as solved by the numerical algorithm, involving the spherical harmonic coefficients of the streamfunctions W and Z and the magnetic potentials C and A. This system of equations requires 12 boundary conditions in order to be well-posed. Since assessing the angular momentum redistribution in our simulations is one of the main goals of this work, we have opted for torque-free velocity and magnetic boundary conditions:

a. impenetrable top and bottom: $v_r = 0|_{r=r_{bot},r_{top}}$,

b. stress free top and bottom: $\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) = \frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) = 0|_{r=r_{bot},r_{top}}$,

c. constant entropy gradient at top and bottom: $\frac{\partial \bar{S}}{\partial r} = cst|_{r=r_{bot},r_{top}}$

d. match to an external potential magnetic field at top and bottom: $\mathbf{B} = \boldsymbol{\nabla}\Phi => \Delta\Phi = 0|_{r=r_{bot},r_{top}}$, or impose a purely radial magnetic field at top and bottom (match to a highly permeable external media, Jackson 1999), i.e $B_\theta = B_\phi = 0|_{r=r_{bot},r_{top}}$

The main difference between having a potential or a purely radial magnetic field is that with the latter the Poynting flux is zero at the shell surface, and thus there is no leakage of magnetic energy through the boundaries.

Appendix A: Model Equations in (r,ℓ,m) space

The anelastic equations (1)–(7) (§2.1) define our physical model. Here we express these equations as they are solved by our numerical algorithm, making use of the velocity and magnetic field decomposition in equations (8)-(9). Diagnostic equations for the streamfunctions and potentials W, Z, A, and C are obtained by considering the vertical component of the momentum and induction equations and the vertical component of their curl. A Poisson equation for pressure can then be derived by taking the divergence of the momentum equation. However, the additional radial derivative this would require can compromise the accuracy of the solution, particularly when applied to the nonlinear advection terms. Since horizontal derivatives are more accurate than vertical derivatives, we choose to only take the horizontal divergence of the momentum equations rather than the full divergence. This results in a diagnostic equation for the horizontal divergence of the velocity field, which is proporational to $\partial W/\partial r$. A spherical harmonic transformation is applied to the governing equations before they are discretized in time so the time stepping occurs in spectral space: (ℓ, m, r). After some manipulation, the governing equations for the spherical harmonic coefficients of the state variables can be expressed as follows:

$$\frac{\ell(\ell+1)}{r^2}\frac{\partial W_{\ell m}}{\partial t} = \mathcal{L}^W + \mathcal{N}^W \tag{1}$$

$$-\frac{\ell(\ell+1)}{r^2}\frac{\partial}{\partial t}\left(\frac{\partial W_{\ell m}}{\partial r}\right) = \mathcal{L}^P + \mathcal{N}^P \tag{2}$$

$$\frac{\ell(\ell+1)}{r^2}\frac{\partial Z_{\ell m}}{\partial t} = \mathcal{L}^Z + \mathcal{N}^Z \tag{3}$$

$$\frac{\partial S_{\ell m}}{\partial t} = \mathcal{L}^S + \mathcal{N}^S. \tag{4}$$

$$\frac{\ell(\ell+1)}{r^2}\frac{\partial A_{\ell m}}{\partial t} = \mathcal{L}^A + \mathcal{N}^A. \tag{5}$$

$$\frac{\ell(\ell+1)}{r^2}\frac{\partial C_{\ell m}}{\partial t} = \mathcal{L}^C + \mathcal{N}^C. \tag{6}$$

In these expressions, the $\mathcal{L}$ denote the linear diffusion, pressure gradient, buoyancy, and volume heating terms which are implemented using a semi-implicit, Crank-Nicolson timestepping method:

$$\mathcal{L}^W = -\frac{\partial P_{\ell m}}{\partial r} - g\rho_{\ell m} + \nu\left(\frac{\ell(\ell+1)}{r^2}\right)\left\{\frac{\partial^2 W_{\ell m}}{\partial r^2} + \left(2\frac{d\ln\nu}{dr} - \frac{1}{3}\frac{d\ln\bar{\rho}}{dr}\right)\frac{\partial W_{\ell m}}{\partial r}\right.$$
$$\left. - \left[\frac{4}{3}\left(\frac{d\ln\nu}{dr}\frac{d\ln\bar{\rho}}{dr} + \frac{d^2\ln\bar{\rho}}{dr^2} + \frac{1}{r}\frac{d\ln\bar{\rho}}{dr} + \frac{3}{r}\frac{d\ln\nu}{dr}\right) + \frac{\ell(\ell+1)}{r^2}\right]W_{\ell m}\right\} ,$$

$$\mathcal{L}^P = \left(\frac{\ell(\ell+1)}{r^2}\right) P_{lm} - \nu \left(\frac{\ell(\ell+1)}{r^2}\right) \left\{ \frac{\partial^3 W_{\ell m}}{\partial r^3} + \left(\frac{d\ln\nu}{dr} - \frac{d\ln\bar{\rho}}{dr}\right) \frac{\partial^2 W_{\ell m}}{\partial r^2} \right.$$
$$- \left(\frac{2}{r}\frac{d\ln\bar{\rho}}{dr} + \frac{d^2\ln\bar{\rho}}{dr^2} + \frac{2}{r}\frac{d\ln\nu}{dr} + \frac{d\ln\nu}{dr}\frac{d\ln\bar{\rho}}{dr} + \frac{\ell(\ell+1)}{r^2}\right) \frac{\partial W_{\ell m}}{\partial r}$$
$$\left. - \left(\frac{d\ln\nu}{dr} + \frac{2}{r} + \frac{2}{3}\frac{d\ln\bar{\rho}}{dr}\right) \frac{\ell(\ell+1)}{r^2} W_{\ell m} \right\} ,$$

$$\mathcal{L}^Z = \nu \left(\frac{\ell(\ell+1)}{r^2}\right) \left\{ \frac{\partial^2 Z_{\ell m}}{\partial r^2} + \left(\frac{d\ln\nu}{dr} - \frac{d\ln\bar{\rho}}{dr}\right) \frac{\partial Z_{\ell m}}{\partial r} \right.$$
$$\left. - \left(\frac{2}{r}\frac{d\ln\nu}{dr} + \frac{d\ln\bar{\rho}}{dr}\,\frac{d\ln\nu}{dr} + \frac{d^2\ln\bar{\rho}}{dr^2} + \frac{2}{r}\frac{d\ln\bar{\rho}}{dr} - \frac{\ell(\ell+1)}{r^2}\right) Z_{\ell m} \right\} ,$$

$$\mathcal{L}^S = -\frac{\ell(\ell+1)}{r^2}\frac{d\bar{S}}{dr} W_{\ell m} + \frac{\kappa_r C_P}{\bar{T}} \left[\frac{\partial^2}{\partial r^2} + \left(\frac{d}{dr}\ln(r^2\kappa_r\bar{\rho})\right)\frac{\partial}{\partial r}\right] (T_{\ell m} + \bar{T}) - \frac{\kappa_r C_P}{\bar{T}}\frac{\ell(\ell+1)}{r^2} T_{\ell m}$$
$$+ \kappa \left[\frac{\partial^2}{\partial r^2} + \left(\frac{d}{dr}\ln(r^2\kappa\bar{\rho}\bar{T})\right)\frac{\partial}{\partial r}\right] (S_{\ell m} + \bar{S}) - \kappa \frac{\ell(\ell+1)}{r^2} S_{\ell m} ,$$

$$\mathcal{L}^A = \eta \left(\frac{\ell(\ell+1)}{r^2}\right) \left\{ \frac{\partial^2 A_{\ell m}}{\partial r^2} + \frac{d\ln\eta}{dr}\frac{\partial A_{\ell m}}{\partial r} - \frac{\ell(\ell+1)}{r^2} A_{\ell m} \right\} ,$$

and

$$\mathcal{L}^C = \eta \left(\frac{\ell(\ell+1)}{r^2}\right) \left\{ \frac{\partial^2 C_{\ell m}}{\partial r^2} - \frac{\ell(\ell+1)}{r^2} C_{\ell m} \right\} .$$

The perfect gas equation of state implies

$$\rho_{\ell m} = \bar{\rho} \left(\frac{1}{\gamma}\frac{P_{\ell m}}{\bar{P}} - \frac{S_{\ell m}}{c_p}\right) \tag{7}$$

and

$$T_{\ell m} = \bar{T} \left(\frac{\gamma - 1}{\gamma}\frac{P_{\ell m}}{\bar{P}} + \frac{S_{\ell m}}{c_p}\right) . \tag{8}$$

The $\mathcal{N}$ terms in equations (A1)–(A6) include nonlinear advection terms which are implemented using an explicit, two-level Adams-Bashforth time stepping method. Although the Coriolis terms are formally linear, they are also included in the $\mathcal{N}$ terms because, unlike the other linear terms, the resulting coefficients depend on azimuthal wavenumber m, and they couple the vertical vorticity equation to the vertical momentum and horizontal divergence equations. This would greatly complicate the matrix solution involved in the Crank-Nicholson method. Thus the $\mathcal{N}$ terms in the momentum equations include Coriolis terms which can be written in spherical harmonic space as:

$$\mathcal{N}^W = \mathcal{A}^W_{\ell m} + \Lambda + \frac{2\Omega_o}{r} \left(\imath m \frac{\partial W_{\ell m}}{\partial r} - (\ell - 1) c^m_\ell Z^m_{\ell-1} + (\ell+2) c^m_{\ell+1} Z^m_{\ell+1} \right) ,$$

$$\mathcal{N}^P = \mathcal{A}^P_{\ell m} + \frac{2\Omega_o}{r^2}\left[-\imath m\left(\frac{\partial W_{\ell m}}{\partial r} + \frac{\ell(\ell+1)}{r}W_{\ell m}\right) + (\ell^2-1)c^m_\ell Z^m_{\ell-1} + \ell(\ell+2)c^m_{\ell+1}Z^m_{\ell+1}\right] ,$$

and

$$\mathcal{N}^Z = \mathcal{A}^Z_{\ell m} + \frac{2\Omega_o}{r^2}\left(-\frac{\ell(\ell^2-1)}{r}c^m_\ell W^m_{\ell-1} + \frac{\ell(\ell+1)(\ell+2)}{r}c^m_{\ell+1}W^m_{\ell+1} + (\ell^2-1)c^m_\ell\frac{\partial W^m_{\ell-1}}{\partial r} + \ell(\ell+2)c^m_{\ell+1}\frac{\partial W^m_{\ell+1}}{\partial r} + \imath m Z_{\ell m}\right) .$$

The $\mathcal{A}^i_{\ell m}$ in these equations represent the spherical harmonic coefficients of the nonlinear velocity advection terms and Lorentz forces. If we define their corresponding configuration space representation as:

$$\mathcal{A}^i(r,\theta,\phi,t) = \sum_{\ell,m}\mathcal{A}^i_{\ell m}(r,t)\, Y_{\ell m}(\theta,\phi) \qquad [i = W,P,Z] , \tag{9}$$

then

$$\mathcal{A}^W = -\bar{\rho}\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) + J_\theta B_\phi - J_\phi B_\theta , \tag{10}$$

$$\mathcal{A}^P = \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta\mathcal{A}_\theta) + \frac{\partial\mathcal{A}_\phi}{\partial\phi}\right\} , \tag{11}$$

and

$$A^Z = \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta\mathcal{A}_\phi) - \frac{\partial\mathcal{A}_\theta}{\partial\phi}\right\} , \tag{12}$$

where

$$\mathcal{A}_\theta = -\bar{\rho}\left(v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{\cos\theta}{r\sin\theta}v_\phi^2\right) + J_\phi B_r - J_r B_\phi \quad , \tag{13}$$

$$\mathcal{A}_\phi = -\bar{\rho}\left(v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{\cos\theta}{r\sin\theta}v_\theta v_\phi\right) + J_r B_\theta - J_\theta B_r \quad , \tag{14}$$

and $\mathbf{J} = \boldsymbol{\nabla}\times\mathbf{B}/(4\pi)$. The dimensonal current density is given by $\mathbf{j} = c\mathbf{J}$.

Likewise, the remaining $\mathcal{N}$ terms represent the spherical harmonic coefficients corresponding to the nonlinear terms in the energy and induction equations:

$$\mathcal{A}^i(r,\theta,\phi,t) = \sum_{\ell,m}\mathcal{N}^i(\ell,m,r,t)\, Y_{\ell m}(\theta,\phi) \qquad [i = S,A,C] , \tag{15}$$

where

$$\mathcal{A}^S = v_r\frac{\partial s}{\partial r} + \frac{v_\theta}{r}\frac{\partial s}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial s}{\partial\phi} + \frac{2\nu}{\bar{T}}\left\{e_{ij}e_{ij} - \frac{1}{3}\left(v_r\frac{d\ln\bar{\rho}}{dr}\right)^2\right\} + \frac{4\pi\eta}{c^2\bar{\rho}\bar{T}}j^2 + \frac{\epsilon}{\bar{T}} \quad , \tag{16}$$

$$\mathcal{A}^A = -\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathcal{E}_r}{\partial\theta}\right) - \frac{1}{r^2\sin^2\theta}\frac{\partial^2\mathcal{E}_r}{\partial\phi^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{r}{\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta\mathcal{E}_\theta) + \frac{\partial\mathcal{E}_\phi}{\partial\phi}\right\}\right] \quad , \tag{17}$$

$$\mathcal{A}^C = \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta\mathcal{E}_\phi) - \frac{\partial\mathcal{E}_\theta}{\partial\phi}\right\} \quad , \tag{18}$$

and $\mathcal{E} = \mathbf{v}\times\mathbf{B}$.

The boundary conditions discussed in §2.1, expressed here in spectral space, require that the boundaries be impenetrable

$$W_{\ell m}(r_{bot}, t) = W_{\ell m}(r_{top}, t) = 0 \quad , \tag{19}$$

and stress-free

$$\frac{\partial^2 W_{\ell m}}{\partial r^2}(r,t) - \left(\frac{2}{r} + \frac{d\ln\bar{\rho}}{dr}\right)\frac{\partial W_{\ell m}}{\partial r}(r,t) = 0 \qquad (r = r_{bot}, r_{top}) \quad , \tag{20}$$

$$\frac{\partial Z_{\ell m}}{\partial r}(r,t) - \left(\frac{2}{r} + \frac{d\ln\bar{\rho}}{dr}\right) Z_{\ell m}(r,t) = 0 \qquad (r = r_{bot}, r_{top}) \quad . \tag{21}$$

We also fix the entropy gradient at the top and bottom boundaries at the value defined by the initial reference state by requiring the perturbation entropy gradient to vanish:

$$\frac{\partial S_{\ell m}}{\partial r}(r_{bot}, t) = \frac{\partial S_{\ell m}}{\partial r}(r_{top}, t) = 0 \quad . \tag{22}$$

The magnetic boundary conditions are chosen such that the interior field is continuous with an external potential field above and below the computational domain:

$$A_{\ell m}(r_{bot}, t) = A_{\ell m}(r_{top}, t) = 0 \quad , \tag{23}$$

$$\frac{\partial C_{\ell m}}{\partial r}(r_{top}, t) + \frac{\ell}{r_{top}} C_{\ell m}(r_{top}, t) = 0 \quad , \frac{\partial C_{\ell m}}{\partial r}(r_{bot}, t) - \frac{\ell+1}{r_{bot}} C_{\ell m}(r_{bot}, t) = 0 \quad . \tag{24}$$

For comparison purposes, we also did several simulations in which the magnetic field was required to be radial at the boundaries, corresponding to a highly permeable external medium (Jackson 1999):

$$\frac{\partial C_{\ell m}}{\partial r}(r,t) = 0 \text{ and } A_{\ell m}(r,t) = 0 \qquad (r = r_{bot}, r_{top}) \quad . \tag{25}$$

Further details on the numerical algorithm are discussed in Clune et al. (1999)