MHD EQUATIONS FOR COMPRESSIBLE CONVECTION IN RAPIDLY ROTATING SPHERICAL SHELLS (ANELASTIC APPROXIMATION)

1 Equation of motion

We first decompose the sensitive thermodynamic variables into the sum of the basic state variables corresponding to the reference atmosphere (here adiabatic) denoted by bar $^{-}$ and the perturbation:

$$\rho = \bar{\rho} + \rho', \qquad P = \bar{P} + p, \qquad S = \bar{S} + s, \qquad T = \bar{T} + \vartheta. \tag{1}$$

The equation of motion can then be written in the following dimensional form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{\nabla p}{\bar{\rho}} + \frac{1}{\bar{\rho}}\mathbf{j}\times\mathbf{B} - 2\mathbf{\Omega}\times\mathbf{u} + \mathbf{F}_\nu + \frac{\rho'\mathbf{g}}{\bar{\rho}}. \tag{2}$$

In order to simplify the numerical method we transform (2) so that the pressure term could be written in form of a gardient $\nabla\left(\frac{p}{\bar{\rho}}\right)$ which can further be cancelled by taking curl of the equation. For the reference adiabatic state we have

$$0 = \nabla\left(\frac{\bar{P}}{\bar{\rho}^\gamma}\right) = \frac{\nabla\bar{P}}{\bar{\rho}^\gamma} - \bar{P}\gamma\bar{\rho}^{-\gamma-1}\nabla\bar{\rho} = \frac{\bar{P}}{\bar{\rho}^\gamma}\left[\frac{\nabla\bar{P}}{\bar{P}} - \frac{\gamma\nabla\bar{\rho}}{\bar{\rho}}\right] = 0.$$

$$\Downarrow$$

$$\nabla\bar{\rho} = \frac{\bar{\rho}\nabla\bar{P}}{\gamma\bar{P}} \tag{3}$$

Using (3) and the hydrostatic balance criterion $\nabla\bar{P} = \bar{\rho}\mathbf{g}$ we can now write $\nabla\left(\frac{p}{\bar{\rho}}\right)$ term as follows:

$$\nabla\left(\frac{p}{\bar{\rho}}\right) = \frac{\nabla p}{\bar{\rho}} - \frac{p}{\bar{\rho}^2}\nabla\bar{\rho} = \frac{\nabla p}{\bar{\rho}} - \frac{p}{\bar{\rho}}\frac{\nabla\bar{P}}{\gamma\bar{P}} = \frac{\nabla p}{\bar{\rho}} - \frac{p}{\gamma\bar{P}}\mathbf{g} \tag{4}$$

now we use the relation $s = c_v\frac{p}{\bar{P}} - c_p\frac{\rho'}{\bar{\rho}}$ and definition $\frac{c_p}{c_v} = \gamma$ to write

$$\frac{p}{\bar{P}} = \frac{s}{c_v} + \gamma\frac{\rho'}{\bar{\rho}} \tag{5}$$

After substituting (5) into (4) we obtain the following relation:

$$\nabla\left(\frac{p}{\bar{\rho}}\right) + \frac{s}{c_p}\mathbf{g} = \frac{\nabla p}{\bar{\rho}} - \frac{\rho'}{\bar{\rho}}\mathbf{g}. \tag{6}$$

From (2) and (6) we obtain the following form of the equation of motion:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\frac{p}{\bar{\rho}} - 2\mathbf{\Omega}\times\mathbf{u} + \frac{1}{\bar{\rho}\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} + \mathbf{F}_\nu + \frac{sg}{c_p}\hat{\mathbf{r}} \tag{7}$$

whith

$$\mathbf{F}_\nu = \nu\left[\frac{1}{\bar{\rho}}\frac{\partial}{\partial x_j}\bar{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{\partial}{\partial x_i}\frac{2}{3\bar{\rho}}\left(\bar{\rho}\frac{\partial u_j}{\partial x_j}\right)\right]$$

where we used $\mu_0\mathbf{j} = \nabla\times\mathbf{B}$.

1.1 Dimensionless equation of motion

To write equation (7) in dimensionless form we use the following definitions:

Dimensionless units

$$
\text{lenght: } d, \quad \text{mass: } \rho_0 d^3
$$

$$
\begin{gathered}
t = \frac{d^2}{\eta} t^*, \quad \mathbf{u} = \frac{\eta}{d}\mathbf{u}^*, \quad \nabla = \frac{1}{d}\nabla^*, \quad \mathbf{\Omega} = \frac{\eta}{d^2}\mathbf{\Omega}^*, \quad \rho = \rho_0 \rho^*, \\
p = \Omega \rho_0 \eta p^*, \quad \mathbf{B} = \sqrt{\Omega \rho_0 \mu_0 \eta}\mathbf{B}^*, \quad \mathbf{F}_\nu = \frac{\nu \eta}{d^3}\mathbf{F}^*_\nu, \quad s = \Delta S S^*, \quad T = T_0 T^*
\end{gathered} \tag{8}
$$

where ρ_0 is density at $\zeta = 1$ (see below) and ΔS is the unit of Entropy. The stared * quantities in (8) represent dimensionless variables. Additionaly, we define the ratio $q \equiv \frac{Pm}{Pr}$ (although it is not necessery). The quanities used to define the dimensionless variables can be combined to define the following dimensionless numbers:

Seven input parameters

$$
Ra = \frac{GMd\Delta S}{\nu \kappa c_p}, \qquad Pr = \frac{\nu}{\kappa}, \qquad Pm = \frac{\nu}{\eta}, \qquad E = \frac{\nu}{2\Omega d^2} \tag{9a-d}
$$

$$
N_\rho = \ln\left(\frac{\rho_i}{\rho_o}\right), \qquad n: \text{ polytropic index}, \qquad \beta = \frac{r_i}{r_o} \tag{9e-g}
$$

Here in dimensional units $G = 6.67 \times 10^{-11}$ is gravitational constant,M is mas of the planet/star, r_o radius of outer shell, r_i radius of inner shell, $d = r_o - r_i$, ΔS is entropy drop from r_i to r_o, ν is eddy kinematic viscosity, assumed constant across shell, κ, η are respectively eddy entropy and magnetic diffusivities, also assumed constant across shell, $\rho_o \equiv \rho(r = r_o)$, $\rho_i \equiv \rho(r = r_i)$ and c_p is specific heat at constant pressure. Ω is rotation rate. For Jupiter we have: $M = 1.9 \times 10^{27}$, $\Omega = 1.76 \times 10^{-4}$, $r_o = 7 \times 10^7$ and $r_i = 5.6 \times 10^7$.

After substituting (8) into (7) and dropping the star subscript * we obtain

$$
\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega} \times \mathbf{u} = Pm\left[-\nabla\left(\frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2\right) - \frac{2}{E}\hat{\mathbf{z}} \times \mathbf{u} + \frac{1}{E\rho}\left(\nabla \times \mathbf{B}\right) \times \mathbf{B} + \mathbf{F}_\nu + q\, Ra \frac{S}{r^2}\hat{\mathbf{r}}\right] \tag{10}
$$

where

$$
\boldsymbol{\omega} = \nabla \times \mathbf{u}, \qquad \mathbf{F}_\nu = \frac{1}{\rho}\frac{\partial}{\partial x_j}\rho\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3\rho}\frac{\partial}{\partial x_i}\rho\frac{\partial u_j}{\partial x_j} \tag{11}
$$

where $\mathbf{u}$ is velocity, S is the entropy and B is the magnetic induction.

We now introduce the following definition of the **polytropic basic state density distribution**:

$$
\begin{gathered}
{}_0 = \frac{2\zeta_0 - \beta - 1}{1-\beta}, \qquad c_1 = \frac{(1+\beta)(1-\zeta_0)}{(1-\beta)^2}, \qquad \zeta_0 = \frac{\beta+1}{\beta \exp\left(N_\rho/n\right) + 1} \\
\bar{\rho}^* = \zeta^n, \qquad \bar{T}^* = \zeta, \qquad \zeta = c_0 + \frac{c_1}{r^*}, \qquad \xi = \frac{n}{\zeta}\frac{d\zeta}{dr}, \qquad r^*_i = \frac{\beta}{1-\beta} < r^* < r^*_o = \frac{1}{1-\beta}
\end{gathered} \tag{12}
$$

where $\bar{T}^* \equiv \frac{\bar{T}}{T_0}$ is the dimensionless basic state adiabatic temperature and $\bar{\rho}^* \equiv \frac{\bar{\rho}}{\rho_0}$ is the dimensionless basic state density. Once again we drop the star superscript for the sake of clarity.
The basic state is programmed in subroutine mes_precompute in module meshs.f90. Both ξ and $d\xi/dr$ are required, and are stored as xi(:,0:1), $d\xi/dr$ being xi(:,1).

Let

$$
\mathbf{F}_\nu = \hat{\mathbf{F}}_\nu - \frac{1}{\bar{\rho}}\nabla \times \nabla \times \bar{\rho}\mathbf{u}, \tag{13}
$$

and

$$\hat{\mathbf{F}}_\nu = \frac{1}{3}\nabla(\nabla\cdot\mathbf{u}) + \frac{1}{\bar\rho}(\nabla\bar\rho\times\boldsymbol{\omega}) - \frac{\mathbf{u}}{\bar\rho}\nabla^2\bar\rho + \frac{2}{3}\hat{\mathbf{r}}u_r\frac{1}{\bar\rho^2}\left(\frac{d\bar\rho}{dr}\right)^2. \tag{14}$$

The first term of this can be ignored since it vanishes when we take the curl of the equation of motion.

$$\frac{1}{\bar\rho}(\nabla\bar\rho\times\boldsymbol{\omega}) = -\xi\omega_\phi\hat{\boldsymbol{\theta}} + \xi\omega_\theta\hat{\boldsymbol{\phi}}, \quad \frac{\mathbf{u}}{\bar\rho}\nabla^2\bar\rho = \mathbf{u}\left(\frac{2}{r}\xi + \frac{d\xi}{dr} + \xi^2\right), \quad \frac{2}{3}\hat{\mathbf{r}}u_r\frac{1}{\bar\rho^2}\left(\frac{d\bar\rho}{dr}\right)^2 = \frac{2}{3}\hat{\mathbf{r}}u_r\xi^2. \tag{15}$$

We write the dimensionless equation of motion in the form

$$\frac{\partial\mathbf{u}}{\partial t} + \frac{1}{\bar\rho}\nabla\times\nabla\times\bar\rho\mathbf{u} = -\nabla\hat{p} + \mathbf{N}_v \tag{16}$$

where

$$\mathbf{N}_v = \mathbf{u}\times\boldsymbol{\omega} + Pm\left[\hat{\mathbf{F}}_\nu - \frac{2}{E}\hat{\mathbf{z}}\times\mathbf{u} + \frac{q\,RaS}{r^2}\hat{\mathbf{r}} + \frac{1}{E\,\rho}(\nabla\times\mathbf{B})\times\mathbf{B}\right], \tag{17}$$

with $\hat{p} \equiv \frac{p}{\bar\rho} + \frac{\mathbf{u}^2}{2}$ and $\hat{\mathbf{F}}_\nu$ being given by (13) and (14). The terms in equation (17) are coded in this order in routine non_velocity in module nonlinear.

2 Continuity equation

In dimensionless units we have

$$\nabla\cdot\bar\rho\mathbf{u} = 0 \tag{18}$$

which implies

$$\mathbf{u} = \frac{1}{\bar\rho}\nabla\times\nabla\times\mathbf{r}\mathcal{P}\bar\rho + \frac{1}{\bar\rho}\nabla\times\mathbf{r}\mathcal{T}\bar\rho, \tag{19}$$

noting carefully that $\mathbf{r}$ and not $\hat{\mathbf{r}}$ appears in equation (19).

3 Entropy equation

We write the dimensional form of the energy equation in terms of the entropy in order to eliminate the temperature completly from our formulation. This is possible if one choses to represent the turbulent heat flux term as being proportional to entropy. For highly turbulent flows the turbulent (eddy) thermal diffusivity κ_T is dominant so we only keep the corresponding heat flux term $c_p\kappa\bar\rho\nabla T$ but written in terms of entropy gradient $\kappa\bar\rho\bar{T}\nabla S$.

$$\bar\rho\bar{T}\frac{\mathcal{D}s}{\mathcal{D}t} = \nabla\cdot\kappa\bar\rho\bar{T} + Q_\nu + Q_j \tag{20}$$

with

$$Q_\nu = \sigma_{ij}\frac{\partial u_i}{\partial x_j}, \qquad \sigma_{ij} = \nu\bar\rho\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}\right) \tag{21}$$

$$Q_j = \eta\mu_0\mathbf{j}^2, \qquad \mu_0\mathbf{j} = \nabla\times\mathbf{B} \tag{22}$$

where $\bar{T}$ is the basic state adiabatic temperature, $\kappa \equiv \kappa_s$ is the eddy entropy diffusivity and $\mathcal{Q}_\nu$ (assumed constant across shell) and $\mathcal{Q}_j$ are respectively viscous and Joule heatings.

3.1 Dimensionless entropy equation

Using definitions (8) we can write (20) as:

$$\frac{\mathcal{D}S}{\mathcal{D}t} = \frac{\kappa}{\eta}\frac{1}{\bar{\rho}\bar{T}}\nabla\cdot\bar{\rho}\bar{T}\nabla S + \frac{\eta\Omega}{T_0\Delta S}\frac{1}{r\bar{h}o\bar{T}}\left(\nabla\times\mathbf{B}\right)^2 + \frac{\nu\eta}{d^2T_0\Delta S}\frac{1}{\bar{T}}\mathcal{Q}^*_\nu \tag{23}$$

with

$$Q^*_\nu = 2\left[e_{ij}e_{ij} - \frac{1}{3}(\nabla\cdot\mathbf{u})^2\right], \qquad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{24}$$

the dimensional units in (23) can be combined using (9a-d) so that

$$\frac{\mathcal{D}S}{\mathcal{D}t} = q\zeta^{-n-1}\nabla\cdot\zeta^{n+1}\nabla S + \frac{Di}{\zeta}\left[E^{-1}\zeta^{-n}\left(\nabla\times\mathbf{B}\right)^2 + Q^*_\nu\right] \tag{25}$$

with

$$Di = \frac{G\,M}{d\,T_0\,c_p}\frac{1}{q\,Ra} = \frac{c_1}{q\,Ra} \tag{26}$$

where we used (12) to eliminate $\bar{\rho}$ and $\bar{T}$. The identification of c_1 parameter in (26) follows from the hydrostatic balance criterion and the equation of state and the polytropic density distribution (12):

$$\left.\begin{array}{l}
\left.\begin{array}{rclcrcl}
\dfrac{dP}{dr} &=& -\overbrace{\dfrac{GM}{r^2}}^{g}\rho & \Rightarrow & \dfrac{P_0(n+1)\zeta^n}{d}\dfrac{d\zeta}{dr^*} &=& -\dfrac{GM}{d^2\,r^{*2}}\rho_0\zeta^n \\
P &=& \rho\,R\,T & \Rightarrow & P_0 &=& \rho_0\,R\,T_0
\end{array}\right\} \Rightarrow R\,T_0\dfrac{n+1}{d}\dfrac{\zeta}{r^*} = -\dfrac{GM}{d^2\,r^{*2}} \\
\hfill r^{*2}\dfrac{d\zeta}{dr^*} = -c_1 \\
\left.\begin{array}{r} n = \gamma - 1 = \frac{c_p}{c_v} - 1 \\ R = c_p - c_v \end{array}\right\} \Rightarrow \dfrac{c_p}{R} = n+1
\end{array}\right\} \Rightarrow c_1 = \frac{GM}{d\,T_0\,c_p}$$

we can write the c_1 coefficient in terms of our input parameters (9e-g)

$$Di = \frac{c_1}{q\,Ra} = \frac{1}{q\,Ra}\frac{\beta(1+\beta)}{(1-\beta)^2}\frac{(\exp(N_\rho/n)-1)}{(1+\beta\exp(N_\rho/n))} \tag{27}$$

Equation (25) is solved by writing

$$\begin{aligned}
\frac{\partial S}{\partial t} - \frac{q}{\rho}\nabla\cdot\rho\nabla S = \frac{\partial S}{\partial t} - q\left[\frac{\partial^2 S}{\partial r^2} - \left(\frac{2}{r} + \xi\right)\frac{\partial S}{\partial r} + \frac{\ell(\ell+1)S}{r^2}\right] = \\
= -\mathbf{u}\cdot\nabla S + q\frac{\xi}{n}\frac{\partial S}{\partial r} + \frac{Di}{\zeta}\left[E^{-1}\zeta^{-n}\left(\nabla\times\mathbf{B}\right)^2 + Q^*_v\right].
\end{aligned} \tag{28}$$

The terms on the right hand side of (28) are programmed into subroutine non_codensity, in module nonlinear.f90. The terms on the left are calculated in tim_lumesh_X and tim_mesh_Y in subroutine timestep.f90. The radial derivative part is computed in subroutine mes_rdom_init in module meshs.f90 and is stored as mes_oc%GreLap. The name is because Entlap is the same as GreLap, see section 7 below. Note that tim_lumesh_X and tim_mesh_Y are set up in subroutine cod_matrices in module codensity.f90, which in turn is called by routine initialise in module main.f90.
Clune et al. gives

$$2\left[e_{ij}e_{ij}-\frac{1}{3}(\nabla\cdot\mathbf{u})^2\right]=2\left(\frac{\partial u_r}{\partial r}\right)^2+2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)^2-\frac{2}{3}\left(\nabla\cdot\mathbf{u}\right)^2+2\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}+\frac{u_\theta\cos\theta}{r\sin\theta}\right)^2+$$
$$\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)^2+\left(\frac{\partial u_\phi}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}\right)^2+\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{u_\phi\cos\theta}{r\sin\theta}\right)^2 \tag{29}$$

Split this into

$$Q_v^*=Q_v^{(1)}+Q_v^{(2)}, \tag{30}$$

where

$$Q_v^{(1)} = 2\left(\frac{\partial u_r}{\partial r}\right)^2+2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)^2-\frac{2}{3}\left(\nabla\cdot\mathbf{u}\right)^2+2\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}+\frac{u_\theta\cos\theta}{r\sin\theta}\right)^2 \tag{31}$$

$$Q_v^{(2)} = \left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)^2+\left(\frac{\partial u_\phi}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}\right)^2+\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{u_\phi\cos\theta}{r\sin\theta}\right)^2 \tag{32}$$

Define

$$q_v^{(1)}=\frac{u_\theta\cos\theta}{r\sin\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi},\qquad q_v^{(2)}=\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{u_\phi\cos\theta}{r\sin\theta}, \tag{33}$$

and we can write

$$Q_v^{(1)}=2\left(\frac{\partial u_r}{\partial r}\right)^2-\frac{2}{3}(\xi u_r)^2+2\left(\frac{\partial u_r}{\partial r}+u_r(\xi+\frac{1}{r})+q_v^{(1)}\right)^2+2\left(q_v^{(1)}+\frac{u_r}{r}\right)^2, \tag{34}$$

and recalling $\boldsymbol{\omega}=\nabla\times\mathbf{u}$,

$$Q_v^{(2)}=\left(2\frac{\partial u_\theta}{\partial r}-\omega_\phi\right)^2+\left(2\frac{\partial u_\phi}{\partial r}+\omega_\theta\right)^2+\left(2q_v^{(2)}+\omega_r\right)^2 \tag{35}$$

Since the components of $\boldsymbol{\omega}$ and $\mathbf{u}$ are all required, evaluated on the physical mesh, for the convective derivative, the only new terms required on the mesh are the radial derivatives of $\mathbf{u}$, $q_v^{(1)}$ and $q_v^{(2)}$. These are all calculated in subroutine tra_qst2rtp_new in module transform.fftw3.f90, which is called by vel_transform in module velocity.f90.

4 Induction equation

The induction equation can be written as follows in dimensional units:

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{u}\times\mathbf{B})+\eta\nabla^2\mathbf{B} \tag{36}$$

using (8) we can write (36) as

Induction equation in dimensionless units

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{u}\times\mathbf{B})+\nabla^2\mathbf{B} \tag{37}$$

and we define

$$\mathbf{N}_B=\nabla\times(\mathbf{u}\times\mathbf{B}) \tag{38}$$

5 Potential formulation

$$\mathbf{u}=\frac{1}{\bar\rho}\nabla\times\nabla\times\mathbf{r}\mathcal{P}\bar\rho+\frac{1}{\bar\rho}\nabla\times\mathbf{r}\mathcal{T}\bar\rho, \tag{39}$$

$$\mathbf{B}=\nabla\times\nabla\times\mathbf{r}\mathcal{P}_B+\nabla\times\mathbf{r}\mathcal{T}_B, \tag{40}$$

5.1 qst form of the variables

We need the qst form of the variables for this evaluation. This is defined by

$$\mathbf{A} = q(r)\mathbf{q} + s(r)\mathbf{s} + t(r)\mathbf{t} \tag{41}$$

where

$$\begin{aligned} \mathbf{q} &= [1,0,0]Y_l^m \qquad (42a) \\ \mathbf{s} &= \frac{1}{\sqrt{l(l+1)}}\left[0, \partial_\theta, \frac{1}{\sin\,\theta}\partial_\phi\right] Y_l^m \qquad (42b) \\ \mathbf{t} &= \frac{1}{\sqrt{l(l+1)}}\left[0, -\frac{1}{\sin\,\theta}\partial_\phi, \partial_\theta\right] Y_l^m \qquad (42c) \end{aligned}$$

$$u_r = q(r)Y_\ell^m, \quad q(r) = \ell(\ell+1)\frac{\mathcal{P}}{r}, \tag{43a – b}$$

$$u_\theta = \frac{1}{\sin\theta}\frac{\partial \mathcal{T}}{\partial\phi} + \frac{1}{r\rho}\frac{\partial}{\partial r}r\rho\frac{\partial\mathcal{P}}{\partial\theta} = -\frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{\sin\theta}\frac{\partial t}{\partial\phi} + \frac{1}{\sqrt{\ell(\ell+1)}}\frac{\partial s}{\partial\theta}, \tag{44}$$

$$u_\phi = -\frac{\partial\mathcal{T}}{\partial\theta} + \frac{1}{r\rho}\frac{\partial}{\partial r}r\rho\frac{\partial\mathcal{P}}{\partial\phi} = \frac{1}{\sqrt{\ell(\ell+1)}}\frac{\partial t}{\partial\theta} + \frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{\sin\theta}\frac{\partial s}{\partial\phi} \tag{45}$$

$$B_r = q_B(r)Y_\ell^m, \quad q_B(r) = \ell(\ell+1)\frac{\mathcal{P}_B}{r}, \tag{46a – b}$$

$$B_\theta = \frac{1}{\sin\theta}\frac{\partial\mathcal{T}_B}{\partial\phi} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\mathcal{P}_B}{\partial\theta} = -\frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{\sin\theta}\frac{\partial t_B}{\partial\phi} + \frac{1}{\sqrt{\ell(\ell+1)}}\frac{\partial s_B}{\partial\theta}, \tag{47}$$

$$B_\phi = -\frac{\partial\mathcal{T}_B}{\partial\theta} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\mathcal{P}_B}{\partial\phi} = \frac{1}{\sqrt{\ell(\ell+1)}}\frac{\partial t_B}{\partial\theta} + \frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{\sin\theta}\frac{\partial s_B}{\partial\phi} \tag{48}$$

Since

$$t = -\sqrt{\ell(\ell+1)}\mathcal{T}, \qquad s = \sqrt{\ell(\ell+1)}\frac{1}{r\rho}\frac{\partial}{\partial r}(r\rho\mathcal{P}) \tag{49a – b}$$

$$t_B = -\sqrt{\ell(\ell+1)}\mathcal{T}_B, \qquad s_B = \sqrt{\ell(\ell+1)}\frac{1}{r}\frac{\partial}{\partial r}(r\mathcal{P}_B) \tag{49c – d}$$

where $\mathcal{P}$ and $\mathcal{T}$ are the poloidal and toroidal scalars defined in (19). Note that q, s and t have these very simple expressions in terms of $\mathcal{P}$ and $\mathcal{T}$, but (49b) involves a radial derivative of the density, so subroutine var_coll_TorPol2qst in module variables.f90 has been modified to add in a compressible term, and a new routine to evaluate the radial derivatives of q,s and t, var_coll_TorPol2qstderiv, has been added to module variables.f90.

The radial derivative terms are exactly analogous to the ordinary terms.

$$q_v^{(1)} = \frac{1}{\sqrt{\ell(\ell+1)}}\frac{\cos\theta}{r\sin\theta}\frac{\partial s}{\partial\theta} + \frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{r\sin^2\theta}\frac{\partial^2 s}{\partial\phi^2} + \frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{r\sin\theta}\frac{\partial^2 t}{\partial\theta\partial\phi} - \frac{1}{\sqrt{\ell(\ell+1)}}\frac{\cos\theta}{r\sin^2\theta}\frac{\partial t}{\partial\phi} \tag{50}$$

$$q_v^{(2)} = \frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{r\sin\theta}\frac{\partial^2 s}{\partial\theta\partial\phi} - \frac{1}{\sqrt{\ell(\ell+1)}}\frac{\cos\theta}{r\sin^2\theta}\frac{\partial s}{\partial\phi} - \frac{1}{\sqrt{\ell(\ell+1)}}\frac{\cos\theta}{r\sin\theta}\frac{\partial t}{\partial\theta} - \frac{1}{\sqrt{\ell(\ell+1)}}\frac{1}{r\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2} \tag{51}$$

5.2 Equations for $\mathcal{P}$, $\mathcal{P}_B$ and $\mathcal{T}$, $\mathcal{T}_B$

$$\frac{\partial \mathcal{T}}{\partial t} - \frac{1}{\rho}\nabla^2(\rho\mathcal{T}) = \frac{\partial \mathcal{T}}{\partial t} - \frac{\partial^2 \mathcal{T}}{\partial r^2} - \left(\frac{2}{r} + 2\xi\right)\frac{\partial \mathcal{T}}{\partial r} - \left(\frac{2\xi}{r} + \xi^2 + \frac{d\xi}{dr}\right)\mathcal{T} + \frac{\ell(\ell+1)\mathcal{T}}{r^2} = \frac{r}{\ell(\ell+1)}\hat{\mathbf{r}}\cdot\nabla\times\mathbf{N}_v \tag{52}$$

$$\frac{\partial \mathcal{P}}{\partial t} - \frac{1}{\rho}\nabla^2(\rho\mathcal{P}) = \frac{\partial \mathcal{P}}{\partial t} - \frac{\partial^2 \mathcal{P}}{\partial r^2} - \left(\frac{2}{r} + 2\xi\right)\frac{\partial \mathcal{P}}{\partial r} - \left(\frac{2\xi}{r} + \xi^2 + \frac{d\xi}{dr}\right)\mathcal{P} + \frac{\ell(\ell+1)\mathcal{P}}{r^2} = \hat{G} \tag{53}$$

$$\nabla^2\hat{G} + \frac{1}{r}\frac{\partial}{\partial r}(r\xi\hat{G}) = \frac{\partial^2 \hat{G}}{\partial r^2} + \left(\frac{2}{r} + \xi\right)\frac{\partial \hat{G}}{\partial r} + \left(\frac{\xi}{r} + \frac{d\xi}{dr}\right)\hat{G} + \frac{\ell(\ell+1)\hat{G}}{r^2} = -\frac{r}{\ell(\ell+1)}\hat{\mathbf{r}}\cdot\nabla\times\nabla\times\mathbf{N}_v \tag{54}$$

and

$$\frac{\partial \mathcal{T}_B}{\partial t} - \frac{1}{\rho}\nabla^2(\rho\mathcal{T}_B) = \frac{\partial \mathcal{T}_B}{\partial t} - \frac{\partial^2 \mathcal{T}_B}{\partial r^2} - \left(\frac{2}{r} + 2\xi\right)\frac{\partial \mathcal{T}_B}{\partial r} - \left(\frac{2\xi}{r} + \xi^2 + \frac{d\xi}{dr}\right)\mathcal{T}_B + \frac{\ell(\ell+1)\mathcal{T}_B}{r^2} = \frac{r}{\ell(\ell+1)}\hat{\mathbf{r}}\cdot\mathbf{N}_B \tag{55}$$

$$\frac{\partial \mathcal{P}_B}{\partial t} - \frac{1}{\rho}\nabla^2(\rho\mathcal{P}_B) = \frac{\partial \mathcal{P}_B}{\partial t} - \frac{\partial^2 \mathcal{P}_B}{\partial r^2} - \left(\frac{2}{r} + 2\xi\right)\frac{\partial \mathcal{P}_B}{\partial r} - \left(\frac{2\xi}{r} + \xi^2 + \frac{d\xi}{dr}\right)\mathcal{P}_B + \frac{\ell(\ell+1)\mathcal{P}_B}{r^2} = \hat{G}_B \tag{56}$$

$$\nabla^2\hat{G}_B + \frac{1}{r}\frac{\partial}{\partial r}(r\xi\hat{G}_B) = \frac{\partial^2 \hat{G}_B}{\partial r^2} + \left(\frac{2}{r} + \xi\right)\frac{\partial \hat{G}_B}{\partial r} + \left(\frac{\xi}{r} + \frac{d\xi}{dr}\right)\hat{G}_B + \frac{\ell(\ell+1)\hat{G}_B}{r^2} = -\frac{r}{\ell(\ell+1)}\hat{\mathbf{r}}\cdot\nabla\times\mathbf{N}_B \tag{57}$$

Here $\hat{G}$, $\hat{G}_B$ are the velocity and magnetic Green's functions for the poloidal equations.
The right hand-sides are computed in subroutine non_velocity in module nonlinear.f90. The curls are computed in subroutines var_coll_qstllcurlr and var_coll_qstllcurlcurlr in module variables.f90. The factor $r/\ell(\ell+1)$ is included in var routines. The terms on the left are calculated in tim_lumesh_vel_X and tim_mesh_vel_Y in subroutine timestep.f90. The radial derivative part is computed in subroutine mes_rdom_init in module meshs.f90 and is stored as mes_oc%CompLap. Note that tim_lumesh_vel_X and tim_mesh_vel_Y are set up in subroutine vel_matrices in module velocity.f90, which in turn is called by routine initialise in module main.f90.

6 Boundary conditions

Entropy

$$S = 1 \quad \text{on} \quad r = r_i, \quad S = 0 \quad \text{on} \quad r = r_o. \tag{58}$$

These boundary conditions are set in subroutine cod_setbc in module codensity.f90.
Toroidal velocity scalar applied on $r = r_i$ or $r = r_o$ as appropriate,

$$\frac{\partial \mathcal{T}}{\partial r} - \frac{\mathcal{T}}{r} = 0 \quad \text{stress}-\text{free}, \quad \mathcal{T} = 0 \quad \text{no}-\text{slip}. \tag{59}$$

Poloidal velocity scalar applied on $r = r_i$ or $r = r_o$ as appropriate,

$$\frac{\partial^2 \mathcal{P}}{\partial r^2} + \xi\frac{\partial \mathcal{P}}{\partial r} = 0 \quad \text{stress}-\text{free}, \quad \frac{\partial \mathcal{P}}{\partial r} = 0 \quad \text{no}-\text{slip}. \tag{60}$$

These assume no-penetration $u_r = 0$ at the boundaries, i.e $P = 0$. This condition is automatically imposed by the Green's function method for solving (54). The velocity boundary conditions are set in vel_bc_Tor and vel_bc_Pol in module velocity.f90.
For the magnetic field at the insulating boundaries, toroidal potential on $r = r_i$ or $r = r_o$ satisfies

$$\mathcal{T}_B = 0 \tag{61}$$

and for the poloidal magnetic potential we have

$$\frac{\partial \mathcal{P}_B}{\partial r} - l\frac{\partial \mathcal{P}_B}{\partial r} = 0 \quad \text{on} \quad r = r_i, \qquad \frac{\partial \mathcal{P}_B}{\partial r} + (l+1)\frac{\partial \mathcal{P}_B}{\partial r} = 0 \quad \text{on} \quad r = r_o. \tag{62}$$

At the interface of a conducting inner core we have

$$[\mathcal{T}_B] = 0, \qquad -\frac{l(l+1)}{r^2}\left[\frac{\partial \mathcal{T}_B}{\partial r}\right] = \frac{1}{r}\hat{\mathbf{r}}\cdot\nabla\times(B_r\,[\mathbf{u}]), \tag{63}$$

$$[\mathcal{P}_B] = 0, \qquad \left[\frac{\partial \mathcal{P}_B}{\partial r}\right] = 0. \tag{64}$$

7 Energy Balance

Multiply (26) by ζ^{n+1} and integrate over the volume of the shell.

$$\frac{\partial}{\partial t}\int \zeta^{n+1} S\,dv = -\int \zeta^{n+1}(\mathbf{u}\cdot\nabla)S\,dv + q\int \nabla\cdot(\zeta^{n+1}\nabla S)\,dv + \int \zeta^n Di Q_v^*\,dv + \int E^{-1} Di\,(\nabla\times\mathbf{B})^2\,dv \tag{65}$$

Now

$$-\int \zeta^{n+1}(\mathbf{u}\cdot\nabla)S\,dv = -\int \nabla\cdot(\zeta^{n+1}\mathbf{u}S)\,dv + \int \nabla\cdot(\zeta^n\mathbf{u})\zeta S\,dv + \int S\zeta^n(\mathbf{u}\cdot\nabla)\zeta\,dv = -c_1\int \zeta^n S\frac{u_r}{r^2}\,dv, \tag{66}$$

using the divergence theorem, the continuity equation (18), and (12). So

$$\begin{aligned}\frac{\partial}{\partial t}\int \zeta^{n+1} S\,dv = -c_1\int \zeta^n S\frac{u_r}{r^2}\,dv - q\int_{S_i}\zeta_i^{n+1}\frac{\partial S}{\partial r}\,r_i^2\sin\theta\,d\theta d\phi + q\int_{S_o}\zeta_o^{n+1}\frac{\partial S}{\partial r}\,r_o^2\sin\theta\,d\theta d\phi \\ + Di\left[\int \zeta^n Q_v^*\,dv + \int E^{-1}\left(\nabla\times\mathbf{B}\right)^2\,dv\right]\end{aligned} \tag{67}$$

Now multiply the equation of motion (10) by $\zeta^n\mathbf{u}\frac{Di}{Pm}$ and integrate over the volume between the shells

$$\frac{Di}{Pm}\frac{\partial}{\partial t}\int \frac{1}{2}\zeta^n\mathbf{u}^2\,dv = \int Di\zeta^n\mathbf{u}\cdot\mathbf{F}_\nu + c_1\int \zeta^n S\frac{u_r}{r^2}\,dv + Di\,E^{-1}\int \mathbf{u}\cdot(\nabla\times\mathbf{B})\times\mathbf{B}\,dv, \tag{68}$$

and using the divergence theorem to remove the pressure term. We multiply the induction equation (37) by $Di\,E^{-1}\,\mathbf{B}$, integrate over the volume between the shells and use the divergence theorem and $\nabla\cdot\mathbf{B} = 0$ to obtain

$$\begin{aligned}Di\,E^{-1}\frac{\partial}{\partial t}\int \mathbf{B}^2\,dv &= Di\,E^{-1}\left[\int \mathbf{B}\cdot\nabla\times(\mathbf{u}\times\mathbf{B})\,dv + \int \mathbf{B}\cdot\nabla^2\mathbf{B}\,dv\right] = \\ &= Di\,E^{-1}\left[\int (\mathbf{u}\times\mathbf{B})\cdot(\nabla\times\mathbf{B})\,dv + \int_{S_i+S_o}[(\mathbf{u}\times\mathbf{B})\times\mathbf{B}]\cdot\hat{\mathbf{n}}\,dS - \int \mathbf{B}\cdot(\nabla\times\nabla\times\mathbf{B})\,dv\right] = \\ &= Di\,E^{-1}\left[\int \mathbf{u}\cdot\mathbf{B}\times(\nabla\times\mathbf{B})\,dv + \int (\mathbf{u}\cdot\mathbf{B})B_r r^2\sin\theta\,d\theta d\phi\Big|_{r_i}^{r_o}\right. \\ &\qquad \left. - \int (\nabla\times\mathbf{B})^2\,dv + \int_{S_i+S_o}\mathbf{B}\times(\nabla\times\mathbf{B})\cdot\hat{\mathbf{n}}\,dS\right]\end{aligned} \tag{69}$$

Now adding (67), (68) and (69) we get the dimensionless global energy balance equation

$$\begin{aligned}\frac{\partial}{\partial t}\int \left(\zeta^{n+1} PrS + \frac{Di}{2Pm}\zeta^n\mathbf{u}^2 + \frac{Di}{E}\mathbf{B}^2\right)dv = \\ \int \left(\zeta^{n+1}\frac{\partial S}{\partial r} + \frac{Di}{E}(\mathbf{u}\cdot\mathbf{B})B_r + \frac{Di}{E}\mathbf{B}\times(\nabla\times\mathbf{B})\cdot\hat{\mathbf{n}}\right) r^2\sin\theta\,d\theta d\phi\Big|_{r_i}^{r_o}\end{aligned} \tag{70}$$

To see that the dissipation terms in (67) and (68) do indeed cancel, note $Di = c_1/qRa$ from (27) and

$$\int \zeta^n \mathbf{u} \cdot \mathbf{F}_\nu \, dv = -\int \zeta^n \frac{\partial u_i}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \zeta^n (\nabla \cdot \mathbf{u})^2 \, dv \tag{71}$$

from (11) and the divergence theorem. Now from (24)

$$2e_{ij}e_{ij} = \frac{\partial u_i}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{72}$$

so that

$$\int \zeta^n Q_v \, dv = -\int \zeta^n \mathbf{u} \cdot \mathbf{F}_\nu \, dv. \tag{73}$$

8 Linearised equations and the Basic State

If $\mathbf{u} = 0$ and S is steady, then (25) implies

$$\nabla \cdot \zeta^{n+1} \nabla \bar{S} = 0, \tag{74}$$

and the solution satisfying the boundary conditions (58) is

$$\bar{S} = \frac{(\zeta_o^{-n} - \zeta^{-n})}{\zeta_o^{-n} - \zeta_i^{-n}}. \tag{75}$$

Note that

$$\zeta_i = c_0 + \frac{c_1}{r_i}, \quad \zeta_o = c_0 + \frac{c_1}{r_o} \tag{76}$$

Writing $S = \bar{S} + S'$, the linear equations are then

$$\frac{\partial S'}{\partial t} = -\mathbf{u} \cdot \nabla \bar{S} + \zeta^{-n-1} \nabla \cdot \zeta^{n+1} \nabla S', \tag{77}$$

or equivalently

$$\frac{\partial S'}{\partial t} = \frac{\xi \zeta^{-n}}{\zeta_o^{-n} - \zeta_i^{-n}} u_r + \zeta^{-n-1} \nabla \cdot \zeta^{n+1} \nabla S', \tag{78}$$

The linearised equation of motion being

$$\frac{\partial \mathbf{u}}{\partial t} = -Pm \left[2E^{-1} \hat{\mathbf{z}} \times \mathbf{u} - \nabla \left(\frac{p'}{\rho} \right) + \mathbf{F}_\nu + \frac{qRaS'}{r^2} \hat{\mathbf{r}} \right]. \tag{79}$$

The induction equation (37) remains unchanged as the nonlinearity is removed through the absence of the Lorentz force in (79):

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \nabla^2 \mathbf{B} \tag{80}$$

8.1 Nusselt number

The Nusselt number is the ratio of the heat conducted in at the bottom divided by the heat conducted in by the basic state. It is also the ratio of the heat conducted out at the top divided by the heat conducted out by the basic state. From the termic part of (70) these must be equal in a steady state, i.e. equal when averaged over time.

$$Nu_{bot} = \frac{\int_{S_i} \zeta_i^{n+1} \frac{\partial S}{\partial r} \, r_i^2 \sin\theta \, d\theta d\phi}{\int_{S_i} \zeta_i^{n+1} \frac{\partial \bar{S}}{\partial r} \, r_i^2 \sin\theta \, d\theta d\phi}, \quad Nu_{top} = \frac{\int_{S_o} \zeta_o^{n+1} \frac{\partial S}{\partial r} \, r_o^2 \sin\theta \, d\theta d\phi}{\int_{S_o} \zeta_o^{n+1} \frac{\partial \bar{S}}{\partial r} \, r_o^2 \sin\theta \, d\theta d\phi}, \tag{81}$$

Using (75), these can be written,

$$Nu_{bot} = -\frac{(\exp N_\rho - 1)\zeta_i r_i^2}{4\pi n c_1} \int_{S_i} \frac{\partial S}{\partial r} \sin\theta \, d\theta d\phi, \tag{82}$$

$$Nu_{top} = -\frac{(1 - \exp - N_\rho)\zeta_o r_o^2}{4\pi n c_1} \int_{S_o} \frac{\partial S}{\partial r} \sin\theta \, d\theta d\phi, \tag{83}$$

or alternatively

$$Nu_{bot} = -\frac{(\exp N_\rho - 1)}{4\pi \xi_i} \int_{S_i} \frac{\partial S}{\partial r} \sin\theta \, d\theta d\phi, \tag{84}$$

$$Nu_{top} = -\frac{(1 - \exp(-N_\rho))}{4\pi \xi_o} \int_{S_o} \frac{\partial S}{\partial r} \sin\theta \, d\theta d\phi, \tag{85}$$

ξ_i and ξ_o being the values on the appropriate boundaries.