First order equations for the Anelastic Approximation

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We define the parameter ϵ measuring the departure from adiabaticity in the usual way:

$$\epsilon = -\frac{d}{T_r}\left[\left(\frac{d\bar{T}}{dr}\right)_r + \frac{g}{c_p}\right] = -\frac{d}{c_p}\left(\frac{d\bar{s}}{dr}\right)_r \ll 1$$

where index r idicates a reference value, bar indicates the reference state, d is the thickness of the system, and assume that the system is described by the perfect gas equation:

$$p = \rho R T \qquad\qquad s = c_v \ln\left(\frac{p}{\rho^\gamma}\right) ,$$

with $\gamma = {}^{c_p}/_{c_v}$. Definig also the nondimensional variables in the following way

$$p = P_r\left(\bar{p} + \epsilon p_1 + \epsilon^2 p_2\right) = g_r \rho_r H_r\left(\bar{p} + \epsilon p_1 + \epsilon^2 p_2\right)$$

$$\rho = \rho_r\left(\bar{\rho} + \epsilon \rho_1 + \epsilon^2 \rho_2\right)$$

$$T = T_r\left(\bar{T} + \epsilon T_1 + \epsilon^2 T_2\right)$$

$$s = s_r + \epsilon c_p\left(\bar{s} + s_1 + \epsilon s_2\right) \quad ^1$$

$$t = \left(\frac{H_r}{\epsilon g}\right)^{1/2} t^* \qquad \mathbf{x} = H_r \mathbf{x}^* \qquad \mathbf{u} = \left(\epsilon g_r H_r\right)^{1/2}\left(\mathbf{u}_0 + \epsilon \mathbf{u}_1\right)$$

and non-dimensional parameters

$$\sigma = \frac{\nu_r}{\kappa_r} \text{ (Prandtl number)} \qquad\qquad Ra = \frac{g_r H_r^3 \epsilon}{\kappa_r \nu_r} \text{ (Rayleigh number)}$$

where H_r is the pressure scale height, and upper star, which will be dropped from now on, denotes non-diomensional variables, leads to the following set of equations:

[1] We bear in mind however that for this expression for the entropy to be valid together with $\bar{s} = \frac{1}{\gamma\epsilon}\ln\left(\frac{\bar{p}}{\bar{\rho}^\gamma}\right)$, also $\ln\left(\frac{\bar{p}}{\bar{\rho}^\gamma}\right) = O(\epsilon)$ must hold, which is a requirement for the basic state (which for a polytrope means that $\frac{1}{\gamma} - \frac{m}{m+1} = O(\epsilon)$, where m is the polytropic index). If this is not the case, we may use a more general expression $s = c_p\left(\bar{s} + \epsilon s_1 + \epsilon^2 s_2\right)$ but we must always hold $\left(\frac{d\bar{s}}{dr}\right) = O(\epsilon)$ in order to use the anelastic approximation.

order ϵ^0 :

$$\bar{\rho}\left[\frac{\partial \mathbf{u}_0}{\partial t} + (\mathbf{u}_0\cdot\nabla)\,\mathbf{u}_0\right] = -\nabla p_1 + \rho_1 \mathbf{g} + \left(\frac{\sigma}{Ra}\right)^{1/2}\nabla\cdot\hat{\tau}_0$$

$$\nabla\cdot(\bar{\rho}\mathbf{u}_0) = 0$$

$$\bar{\rho}\bar{T}\left[\frac{\partial s_1}{\partial t} + \mathbf{u}_0\cdot\nabla\left(\bar{s}+s_1\right)\right] = \left(\frac{1}{Ra\sigma}\right)^{1/2}\nabla\cdot\left[\bar{T}\nabla\left(\bar{s}+s_1\right)\right] + \frac{1}{4}\frac{g_r H_r}{c_p T_r}\left(\frac{\sigma}{Ra}\right)^{1/2}\hat{\tau}_0 : \hat{\tau}_0$$

$$\frac{p_1}{\bar{p}} = \frac{T_1}{\bar{T}} + \frac{\rho_1}{\bar{\rho}} \qquad s_1 = \frac{1}{\gamma}\frac{p_1}{\bar{p}} - \frac{\rho_1}{\bar{\rho}}$$

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order ϵ^1 :

$$\bar{\rho}\left[\frac{\partial \mathbf{u}_1}{\partial t} + (\mathbf{u}_1\cdot\nabla)\,\mathbf{u}_0 + (\mathbf{u}_0\cdot\nabla)\,\mathbf{u}_1\right] = -\rho_1\left[\frac{\partial \mathbf{u}_0}{\partial t} + (\mathbf{u}_0\cdot\nabla)\,\mathbf{u}_0\right] - \nabla p_2 + \rho_2\mathbf{g} + \left(\frac{\sigma}{Ra}\right)^{1/2}\nabla\cdot\hat{\tau}_1$$

$$\frac{\partial \rho_1}{\partial t} + \nabla\cdot\left(\bar{\rho}\mathbf{u}_1 + \rho_1\mathbf{u}_0\right) = 0$$

$$\begin{aligned}\bar{\rho}\bar{T}\left[\frac{\partial s_2}{\partial t} + \mathbf{u}_0\cdot\nabla s_2 + \mathbf{u}_1\cdot\nabla\left(\bar{s}+s_1\right)\right] &= -\left(\bar{\rho}T_1 + \rho_1\bar{T}\right)\left[\frac{\partial s_1}{\partial t} + \mathbf{u}_0\cdot\nabla\left(\bar{s}+s_1\right)\right] + \\ &+ \left(\frac{1}{Ra\sigma}\right)^{1/2}\nabla\cdot\left[\bar{T}\nabla s_2 + T_1\nabla\left(\bar{s}+s_1\right)\right] + \frac{1}{2}\frac{g_r H_r}{c_p T_r}\left(\frac{\sigma}{Ra}\right)^{1/2}\hat{\tau}_0 : \hat{\tau}_1\end{aligned}$$

$$\frac{p_2}{\bar{p}} = \frac{T_2}{\bar{T}} + \frac{\rho_2}{\bar{\rho}} + \frac{\rho_1 T_1}{\bar{\rho}\bar{T}} \qquad s_2 = \frac{1}{\gamma}\frac{p_2}{\bar{p}} - \frac{\rho_2}{\bar{\rho}} - \frac{1}{\gamma}\left(\frac{p_1}{\bar{p}}\right)^2 + \left(\frac{\rho_1}{\bar{\rho}}\right)^2$$

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where $\tau_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\nabla\cdot\mathbf{u}\delta_{ij}$ is the stress tensor (here the indexes $i, j = 1, 2, 3$ number the components), and since $\tau(\mathbf{u})$ is a linear operator in $\mathbf{u}$, $\tau_0 = \tau(\mathbf{u}_0)$ and $\tau_1 = \tau(\mathbf{u}_1)$. The double dot means $\tau : \tau \doteq \tau_{ij}\tau_{ij} = \tau_{ij}\tau_{ji} = \mathrm{Tr}\left(\tau^2\right)$. The entropy formulation was used with the heat flux expressed in terms of entropy as $T\nabla s$.

Normally, of course, $(\mathbf{u}_0, s_1, p_1, \rho_1, T_1)$ are calculated from the zero-order equations and then are used to calculate the first order corrections $(\mathbf{u}_1, s_2, p_2, \rho_2, T_2)$.